1. Proof of Lemma 4.3

For $k \geq 0$, $(T1^k(T_u), T2^k(T_u), T3^k(T_u))$ is the triad obtained from optimal feasible solution $X^k(T_u)$ of problem $(TMP3)^k(T_u)$.

By construction of $(TMP3)^k(T_u)$, it is clear that $T1^k(T_u) < T_u, \forall k \geq 1$.

In order to prove relation (4), note that for a triad $(T1^k(T_u), T2^k(T_u), T3^k(T_u))$, $k \geq 0$, the next triad $(T1^{k+1}(T_u), T2^{k+1}(T_u), T3^{k+1}(T_u))$ is obtained from an optimal feasible solution $X^{k+1}(T_u)$ of TMTP $(TMP3)^{k+1}(T_u)$ or $(TMP3)^{k(\delta^*)}(T_u)$ for some value of $\delta = \delta^*$.

If $T2^k(T_u) = t_2^{\beta_k}$, $1 \leq \beta_k \leq \beta$, then the problem $(TMP3)^{k+1}(T_u)$ is

$$(TMP3)^{k+1}(T_u) \qquad \min_{X \in \hat{S}} \max_{I_O \times I_T} (t'_{ij} : x_{ij} > 0)$$

where
$$t'_{ij} = \begin{cases} \left. \begin{matrix} 0 & \text{if } t_{ij} \leq T_u \\ M & \text{if } t_{ij} \geq T_u \end{matrix} \right\} & \text{for } (i,j) \in P_1 \\ \left. \begin{matrix} 0 & \text{if } t_{ij} \leq t_2^{\beta_k - \delta^*} \\ M & \text{if } t_{ij} > t_2^{\beta_k - \delta^*} \end{matrix} \right\} & \text{for } (i,j) \in P_2 \\ \left. \begin{matrix} t_{ij} & \text{if } t_{ij} < T2^k(T_u) + T3^k(T_u) - t_2^{\beta_k - \delta^*} \\ M & \text{if } t_{ij} \geq T2^k(T_u) + T3^k(T_u) - t_2^{\beta_k - \delta^*} \end{matrix} \right\} & \text{for } (i,j) \in P_3 \end{cases}$$

and $\hat{S}$ is same as defined in (2).

Clearly, $T2^{k+1}(T_u) \leq t_2^{\beta_k - \delta^*} < t_2^{\beta_k} = T2^k(T_u), \forall \ k \geq 1$.

$$\Rightarrow \ T2^{k+1}(T_u) < T2^k(T_u) \ \forall \ k \geq 0$$

which proves the relation (4).

Next, for proving the relation (5), suppose $T3^{k+1}(T_u) < T3^k(T_u)$ for some $k = \bar{k} \in \{0, 1, 2, ..., s-1\}$, i.e., $T3(\bar{k}+1) < T3(\bar{k})$ where $\bar{k} \in \{1, 2, ..., s-1\}$.

Then $X(\bar{k}+1)$ yields the triad $(T1(\bar{k}+1), T2(\bar{k}+1), T3(\bar{k}+1))$ having $T1(\bar{k}+1) < T_u$, $T2(\bar{k}+1) < T2(\bar{k})$ and $T3(\bar{k}+1) < T3(\bar{k})$.

Therefore, by construction of $(TMP3)^{\bar{k}}(T_u)$, it follows that $X(\bar{k}+1)$ is a restricted feasible solution of the problem $(TMP3)^{\bar{k}}(T_u)$ yielding Phase-3 time as $T3(\bar{k}+1) < T3(\bar{k})$, a contradiction to the optimality of $X(\bar{k})$ (yielding Phase-3 time as $T3(\bar{k})$) for the problem $(TMP3)^{\bar{k}}(T_u)$.

Hence, $T3^{k+1}(T_u) \geq T3^k(T_u) \ \forall \ k \geq 1$ and the relation (5) follows.

Next, for any $k \in \{1, 2, ..., s-1\}$,

$$T2^{k+1}(T_u) \leq t_2^{\beta_k - \delta^*}$$
$$\text{and } T3^{k+1}(T_u) < T2^k(T_u) + T3^k(T_u) - t_2^{\beta_k - \delta^*}$$
$$\Rightarrow \ T2^{k+1}(T_u) + T3^{k+1}(T_u) < T2^k(T_u) + T3^k(T_u) \ \forall \ k \geq 1$$

and hence, the relation (6) also follows.

2. Proof of Theorem 4.4

Suppose there exist a feasible solution X^* of 3-PhTP with corresponding triad $(T1^*, T2^*, T3^*)$ such that $T2^* + T3^* < T2^s(T_u) + T3^s(T_u)$ and $T1^* < T_u$.

In view of relation (4) of lemma 4.3,

$$T2(0) > T2(1) > T2(2) > \cdots > T2^s(T_u)$$

Three cases arise.

Case 1. $T2^* > T2(0)$.
Since, $T2^* + T3^* < T2^s(T_u) + T3^s(T_u) < T2(0) + T3(0)$ (using Lemma 4.3), it follows that $T3^* < T3(0)$.
Also $T1^* < T_u$.
X^* is a restricted feasible solution of the problem $(TMP3)^0(T_u)$ yielding optimal value as $T3^* < T3(0)$. But $X(0)$ is an optimal feasible solution of TMTP $(TMP3)^0(T_u)$, a contradiction.

Case 2. $T2^* < T2^s(T_u)$.
If $T2^s(T_u) = T2^{min}$, then $T2^* < T2^s(T_u)$ is not possible, as $T2^{min}$ is the minimum time of Phase-2.
Therefore $T2^s(T_u) > T2^{min}$.
Since $T1^* < T_u$, $T2^* < T2^s(T_u)$ and $T2^* + T3^* < T2^s(T_u) + T3^s(T_u)$, it follows that X^* must be a restricted feasible solution of one of the problems of the collection $(TMP3)^{s(\delta)}(T_u)$, $\delta = 1, 2, ...,.$
But this is a contradiction as $(T1^s(T_u), T2^s(T_u), T3^s(T_u))$ is the last triad of the sequence and none of the problems of the collection $(TMP3)^{s(\delta)}(T_u)$ is restricted feasible.

Case 3. $T2^s(T_u) \leq T2^* \leq T2(0)$.

In this case, two possibilities arise.

Possibility (a).
$$T2^* = T2(\bar{k}) \text{ for some } \bar{k} \in \{0, 1, 2, ..., s\}$$
$$\text{Since, } T2^* + T3^* < T2^s(T_u) + T3^s(T_u) < T2(\bar{k}) + T3(\bar{k})$$
$$\Rightarrow \quad T3^* < T3(\bar{k}) < T2(\bar{k} - 1) + T3(\bar{k} - 1) - t_2^{\beta_{\bar{k}-1} - \delta^*}$$
$$\text{Also, } T2^* = T2(\bar{k}) \leq t_2^{\beta_{\bar{k}-1} - \delta^*} \text{ and } T1^* < T_u$$

$\Rightarrow X^*$ is a restricted feasible solution of TMTP $(TMP3)^{\bar{k}}(T_u)$ providing objective function value as $T3^* < T3(\bar{k})$, a contradiction to the fact that $X(\bar{k})$ is an optimal feasible solution for the TMTP $(TMP3)^{\bar{k}}(T_u)$.

Possibility (b).
$$T2(\bar{k} + 1) < T2^* < T2(\bar{k}) \text{ for some } \bar{k} \in \{0, 1, 2, ..., s - 1\}$$
$$\text{Since } T2^* + T3^* < T2^s(T_u) + T3^s(T_u) \leq T2(\bar{k} + 1) + T3(\bar{k} + 1)$$
$$\Rightarrow \quad T3^* < T3(\bar{k} + 1) \quad (\text{as } T2^* > T2(\bar{k} + 1))$$

Now $(T1(\bar{k}+1), T2(\bar{k}+1), T3(\bar{k}+1))$ is a triad of Phase-1, Phase-2 and Phase-3 shipment times yielded from an optimal solution $X(\bar{k} + 1)$ of TMTP $(TMP3)^{\bar{k}+1}(T_u)$

which is the first restricted feasible problem $(TMP3)^{\bar{k}(\delta^*)}(T_u)$ in the collection $(TMP3)^{\bar{k}(\delta)}(T_u)$ of problems solved successively by taking values of $\delta = 1, 2, ..$ and so on.

Further $T2(\bar{k} + 1) < T2^* < T2(\bar{k})$,

$T2^* + T3^* < T2(\bar{k}) + T3(\bar{k})$,

$T1^* < T_u$

$\Rightarrow X^*$ must be a restricted feasible solution of one of the problems $(TMP3)^{\bar{k}(\delta)}(T_u)$ for $\delta = 1, 2, ..., \delta^*$, i,e., X^* must be a restricted feasible solution of one of the problems $(TMP3)^{\bar{k}(1)}(T_u)$ to $(TMP3)^{\bar{k}(\delta^*)}(T_u)$.

X^* cannot be a restricted feasible solution of any of the problems $(TMP3)^{\bar{k}(1)}(T_u)$ to $(TMP3)^{\bar{k}(\delta^*-1)}(T_u)$, as $(TMP3)^{\bar{k}(\delta^*)}(T_u)$ is the first restricted feasible problem among the problems $(TMP)_3^{\bar{k}(\delta)}(T_u)$.

Therefore, X^* must be a restricted feasible solution of the problem $(TMP3)^{\bar{k}(\delta^*)}(T_u)$ yielding triad $(T1^*, T2^*, T3^*)$ with $T3^* < T3(\bar{k} + 1)$, which is a contradiction to the optimality of $X(\bar{k} + 1)$ for the problem $(TMP3)^{\bar{k}(\delta^*)}(T_u)$.

3. Proof of Lemma 4.5

Since $(T1^{prof}(l + 1), T2^{prof}(l + 1), T3^{prof}(l + 1))$ is a proficient triad with respect to Phase-1 under the restriction that Phase-1 time is strictly less than $T1^{prof}(l)$, $\forall\, l = 1, 2, ..., v$, therefore

$$T1^{prof}(l) > T1^{prof}(l + 1) \,\forall\, l \geq 0$$

and hence the relations (7) follow.

In order to prove relation (8), let, if possible, there exist $\bar{l} \in \{0, 1, 2, ..., v\}$ such that

$$T2^{prof}(\bar{l} + 1) + T3^{prof}(\bar{l} + 1) < T2^{prof}(\bar{l}) + T3^{prof}(\bar{l})$$

Note that $\quad T1^{prof}(\bar{l} + 1) < T1^{prof}(\bar{l}) < T1^{prof}(\bar{l} - 1)$

This contradicts the fact that $T2^{prof}(\bar{l}) + T3^{prof}(\bar{l})$ is the minimum sum of Phase-2 and Phase-3 shipment times under the restriction that Phase-1 time is strictly less than $T1^{prof}(\bar{l} - 1)$ (Ref. Theorem 4.4).

4. Proof of Theorem 4.6

Let, if possible, there exist a feasible solution X^* of 3-PhTP yielding the triad $(T1^*, T2^*, T3^*)$ with $T1^* + T2^* + T3^* < \min\limits_{\{l=1,2,...,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right]$.

Then, depending on the value of $T1^*$, following three cases arise:

Case 1. $T1^* > T1^{prof}(0)$

$$T1^* + T2^* + T3^* < \min_{\{l=1,2,...,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right]$$

$$\Rightarrow T1^* + T2^* + T3^* < T1^{prof}(0) + T2^{prof}(0) + T3^{prof}(0)$$

Since $T1^* > T1^{prof}(0) \Rightarrow T2^* + T3^* < T2^{prof}(0) + T3^{prof}(0)$

This implies that X^* is a feasible solution of 3-PhTP yielding sum of Phase-2 and Phase-3 shipment times strictly less than $T2^{prof}(0) + T3^{prof}(0)$, which is a contradiction, as $T2^{prof}(0) + T3^{prof}(0)$ is the minimum sum of shipment times of Phase-2 and Phase-3 (Ref. Theorem 4.4 and Remark 3).

Case 2. $T1^* < T1^{prof}(v)$

From Algorithm, it is clear that $T1^{prof}(v) = T1^{min}$.
Therefore, $T1^* < T1^{prof}(v)$ is not possible.

Case 3. $T1^{prof}(v) \le T1^* \le T1^{prof}(0)$
Two sub cases arise:

Sub case (i) $T1^* = T1^{prof}(\bar{l})$ for some $\bar{l} \in \{0, 1, 2, ..., v\}$.

$$\text{Since } T1^* + T2^* + T3^* < \min_{\{l=1,2,...,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right]$$

$$\Rightarrow T1^* + T2^* + T3^* < T1^{prof}(\bar{l}) + T2^{prof}(\bar{l}) + T3^{prof}(\bar{l})$$

$$\Rightarrow T2^* + T3^* < T2^{prof}(\bar{l}) + T3^{prof}(\bar{l})$$

This implies that X^* is a feasible solution of 3-PhTP yielding sum of shipment times of Phase-2 and Phase-3 strictly less than $T2^{prof}(\bar{l}) + T3^{prof}(\bar{l})$ corresponding to Phase-1 time $T1^* = T1^{prof}(\bar{l})$, which is a contradiction to the fact that $T2^{prof}(\bar{l}) + T3^{prof}(\bar{l})$ is the minimum sum of shipment times of Phase-2 and Phase-3 corresponding to Phase-1 time $T1^{prof}(\bar{l})$ because of $(T1^{prof}(\bar{l}), T2^{prof}(\bar{l}), T3^{prof}(\bar{l})$ being a proficient triad with respect to Phase-1.

Sub case (ii) $T1^{prof}(\bar{l}+1) < T1^* < T1^{prof}(\bar{l})$ for some $\bar{l} \in \{1, 2, ..., v-1\}$.

$$\text{Since } T1^* + T2^* + T3^* < \min_{\{l=1,2,...,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right]$$

$$\Rightarrow T1^* + T2^* + T3^* < T1^{prof}(\bar{l}+1) + T2^{prof}(\bar{l}+1) + T3^{prof}(\bar{l}+1)$$

$$\Rightarrow T2^* + T3^* < T2^{prof}(\bar{l}+1) + T3^{prof}(\bar{l}+1) \text{ (as } T1^* > T1^{prof}(\bar{l}+1))$$

Also $T1^* < T1^{prof}(\bar{l})$.
But $T2^{prof}(\bar{l}+1) + T3^{prof}(\bar{l}+1)$ is the minimum sum of shipment times of Phase-2 and Phase-3 under the restriction that Phase-1 time is strictly less than $T1^{prof}(\bar{l})$, which is a contradiction.

5. Numerical Illustration

Example 1. Consider a balanced 3-PhTP (namely 3-PhTP(ex1) as in Table 1) with number of origins as $m = 6$ and number of terminals as $n = 8$ in the form of time matrix specifying the time of transportation of a homogeneous commodity via each origin-terminal link. The cells with underlined entries denote Phase-1 links, cells with shaded entries denote Phase-2 links and remaining cells correspond to Phase-3 links.

Table 1. 3-PhTP(ex1)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	06	03	12	21	15	32	08	42	15
S_2	40	04	29	03	23	17	03	06	10
S_3	12	21	02	07	08	11	13	24	12
S_4	08	05	09	04	09	15	37	09	06
S_5	29	27	17	19	08	14	15	23	17
S_6	11	35	42	05	27	32	35	27	20
b_j	12	05	13	06	09	20	08	07	

Solution. From Table 1, the parameters P_1, P_2 and P_3 are noted as follows

$$P_1 = \left\{ \begin{array}{l} (1,1),\ (1,4),\ (2,2),\ (2,5),\ (2,6),\ (2,8),\ (3,7),\ (4,1),\ (4,3), \\ (4,4),(4,6),\ (5,5),\ (5,8),\ (6,2),\ (6,5),\ (6,7) \end{array} \right\}$$

$$P_2 = \left\{ \begin{array}{l} (1,2),\ (1,3),\ (1,5),\ (2,7),\ (3,1),\ (3,3),\ (3,4),\ (3,6),\ (4,5), \\ (4,8),\ (5,2),\ (5,3),\ (5,7),\ (6,6),\ (6,8) \end{array} \right\}$$

$$P_3 = \left\{ \begin{array}{l} (1,6),\ (1,7),\ (1,8),\ (2,1),\ (2,3),\ (2,4),\ (3,2),\ (3,5),\ (3,8), \\ (4,2),(4,7),\ (5,1),\ (5,4),\ (5,6),\ (6,1),\ (6,3),\ (6,4) \end{array} \right\}$$

The distinct time entries in three phases can be arranged in increasing order as follows.

Phase-1: $t_1^1(= 04) < t_1^2(= 06) < t_1^3(= 08) < t_1^4(= 09) < t_1^5(= 13) < t_1^6(= 15) < t_1^7(= 17) < t_1^8(= 21) < t_1^9(= 23) < t_1^{10}(= 27) < t_1^{11}(= 35) = t_1^\alpha;\ \alpha = 11$

Phase-2: $t_2^1(= 02) < t_2^2(= 03) < t_2^3(= 07) < t_2^4(= 09) < t_2^5(= 11) < t_2^6(= 12) < t_2^7(= 15) < t_2^8(= 17) < t_2^9(= 27) < t_2^{10}(= 32) = t_2^\beta;\ \beta = 10$

Phase-3: $t_3^1(= 03) < t_3^2(= 05) < t_3^3(= 08) < t_3^4(= 11) < t_3^5(= 14) < t_3^6(= 19) < t_3^7(= 21) < t_3^8(= 24) < t_3^9(= 29) < t_3^{10}(= 32) < t_3^{11}(= 37) < t_3^{12}(= 40) < t_3^{13}(= 42);\ \gamma = 13$

Solving 3-PhTP(ex1) by generating all proficient triads with respect to Phase-1:

Step 1. On solving the standard TMTPs $(TMP1)^0$ and $(TMP2)^0$ (Table 2 and 3), the minimum shipment times of Phase-1 and Phase-2 are obtained as follows.

$$T1^{min} = 0 \text{ and } T2^{min} = 0$$

Table 2.TMTP$(TMP1)^0$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1		(0)	(0)			(7)	(3)		15
	06	00	00	21	00	00	00	00	
S_2			(10)						10
	00	04	00	00	23	17	00	06	
S_3			(3)	(6)	(3)				12
	00	00	00	00	00	00	13	00	
S_4				(6)					06
	08	00	09	04	00	15	00	00	
S_5	(12)						(5)		17
	00	00	00	00	08	00	00	23	
S_6						(13)	(7)		20
	00	35	00	00	27	00	35	00	
b_j	12	05	13	06	09	20	08	07	

Set $t_1^0 = t_2^0 = t_3^0 = 0$, $T_1^{prof}(0) = \infty$ and go to Step 2 for $l = 0$.

Step 2. $T1^{prof}(0) = \infty > T1^{min} = 0$, therefore go to Step 3.

Step 3. Set $T1^{prof}(0) = \infty = T_u$. Solving the TMTP $(TMP3)^0(T_u = \infty)$ (which is same as $(TMP3)^0$), obtain an optimal feasible solution $X^0(T_u = \infty)$ of problem $(TMP3)^0(T_u = \infty)$ (Table 4) and corresponding optimal time as $T3^0(T_u = \infty) = 0$. Obtain the corresponding Phase-1 and Phase-2 times as $T1^0(T_u = \infty) = 35$ and $T2^0(T_u = \infty) = 32$ respectively so that the first triad is $(T1^0(T_u = \infty), T2^0(T_u = \infty), T3^0(T_u = \infty)) = (35, 32, 00)$.
Therefore, go to Step 4 for $k = 0$.

Step 4. $T2^0(T_u) = 32 > T2^{min}$. Find β_0, $1 \le \beta_0 \le \beta$ such that $T2^0(T_u) = 32 = t_2^{10}$. This implies that $\beta_0 = 10$. Next, go to Step 5 for $\delta = 1$.

Step 5. On solving the problem $(TMP3)^{0(1)}(T_u = \infty)$ (Table 5), it comes out to be restricted feasible, therefore, proceed to Step 6.

Table 3.TMTP$(TMP2)^0$

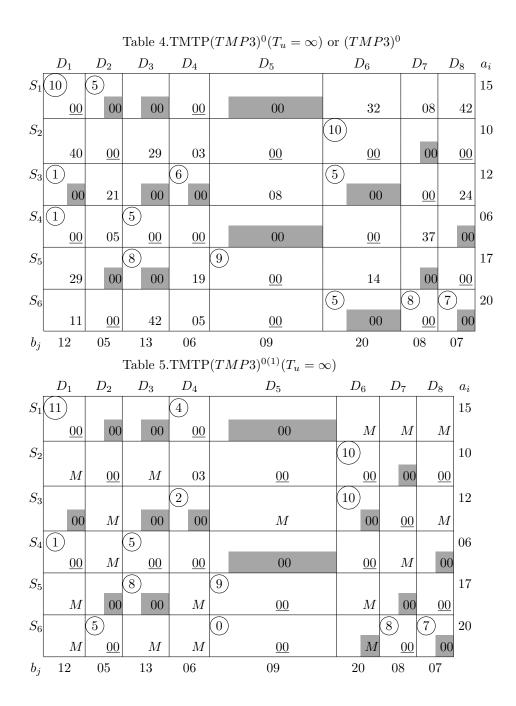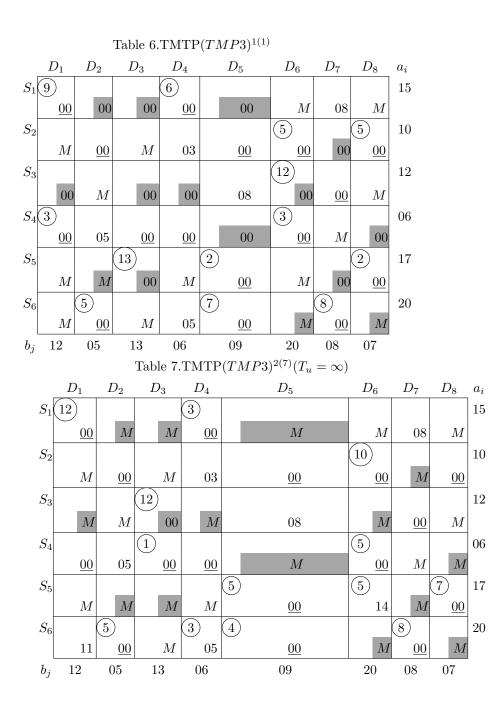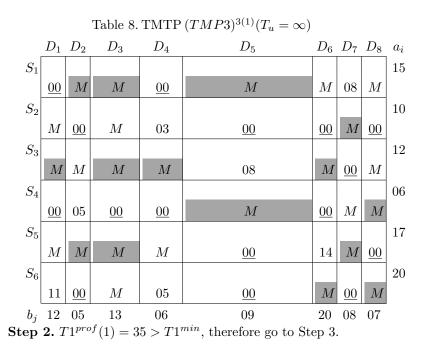|	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	⑫			②		①			15
	$\underline{00}$	03	12	$\underline{00}$	15	00	00	00	
S_2			⑩						10
	00	$\underline{00}$	00	00	$\underline{00}$	$\underline{00}$	03	$\underline{00}$	
S_3				⑨			③		12
	12	00	02	07	00	11	$\underline{00}$	00	
S_4						⑥			06
	$\underline{00}$	00	$\underline{00}$	$\underline{00}$	09	$\underline{00}$	00	09	
S_5						⑬		④	17
	00	27	17	00	$\underline{00}$	00	15	$\underline{00}$	
S_6		⑤	③	④			⑧		20
	00	$\underline{00}$	00	00	$\underline{00}$	32	$\underline{00}$	27	
b_j	12	05	13	06	09	20	08	07	

Step 6. Here $\delta = 1 = \delta^*$ and the problem $(TMP3)^{0(1)}(T_u = \infty)$ is $(TMP3)^1(T_u = \infty)$ whose optimal feasible solution yields the next triad $(T1^1(T_u), T2^1(T_u), T3^1(T_u)) = (35, 27, 00)$. Next repeating Step 3 to Step 6 for $(T1^1(T_u), T2^1(T_u), T3^1(T_u)) = (35, 27, 00)$, the next triad $(T1^2(T_u), T2^2(T_u), T3^3(T_u)) = (35, 17, 00)$ is obtained from problem $(TMP3)^{1(1)}(T_u)$ (or $(TMP3)^2(T_u)$ renamed, Table 6). Further, on solving problems of collection $(TMP3)^{2(\delta)}(T_u)$, $\delta = 1, 2, ...$ successively, the first restricted feasible problem comes out to be $(TMP3)^{2(7)}(T_u)$ (or $(TMP3)^3(T_u)$ renamed, Table 7) which yields the next triad $(T1^3(T_u), T2^3(T_u), T3^3(T_u)) = (35, 2, 14)$. Next, for the triad $(T1^3(T_u), T2^3(T_u), T3^3(T_u)) = (35, 2, 14)$, where $T2^3(T_u) = t_2^{\alpha_3} = 2 = t_2^1$, the corresponding problem $(TMP3)^{3(1)}(T_u)$ (Table 8) is not restricted feasible. Also $t_2^{\alpha_3 - 1} = t_2^0 = 0 = T2^{min}$, therefore Step 7 follows which concludes that the current triad $(T1^3(T_u), T2^3(T_u), T3^3(T_u)) = (35, 2, 14)$ is the last triad of the above sequence of triads and can be used as an initial proficient triad with respect to Phase-1 i.e., $(T1^{prof}(1), T2^{prof}(1), T3^{prof}(1)) = (35, 2, 14)$.

Next, for finding a proficient triad with respect to Phase-1 under the restriction that Phase-1 time is strictly less than $T1^{prof}(1) = 35$, proceed to Step 2 for $l = 1$ and obtain a sequence of triads, the last of which will be the required proficient triad.

Table 4.TMTP$(TMP3)^0$ $(T_u = \infty)$ or $(TMP3)^0$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	⑩ 00	⑤ 00	00	00	00	32	08	42	15
S_2	40	00	29	03	00	⑩ 00	00	00	10
S_3	① 00	21	00	⑥ 00	08	⑤ 00	00	24	12
S_4	① 00	05	⑤ 00	00	00	00	37	00	06
S_5	29	⑧ 00	00	⑨ 19	00	14	00	00	17
S_6	11	00	42	05	00	⑤ 00	⑧ 00	⑦ 00	20
b_j	12	05	13	06	09	20	08	07	

Table 5.TMTP$(TMP3)^{0(1)}$ $(T_u = \infty)$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	⑪ 00	00	00	④ 00	00	M	M	M	15
S_2	M	00	M	03	00	⑩ 00	00	00	10
S_3	00	M	00	② 00	M	⑩ 00	00	M	12
S_4	① 00	M	⑤ 00	00	00	00	M	00	06
S_5	M	00	⑧ 00	M	⑨ 00	M	00	00	17
S_6	M	⑤ 00	M	⓪ M	00	M	⑧ 00	⑦ 00	20
b_j	12	05	13	06	09	20	08	07	

Table 6.TMTP$(TMP3)^{1(1)}$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	⑨			⑥					15
	00	00	00	00	00	M	08	M	
S_2						⑤		⑤	10
	M	00	M	03	00	00	00	00	
S_3						⑫			12
	00	M	00	00	08	00	00	M	
S_4	③					③			06
	00	05	00	00	00	00	M	00	
S_5			⑬	②				②	17
	M	M	00	M	00	M	00	00	
S_6		⑤		⑦			⑧		20
	M	00	M	05	00	M	00	M	
b_j	12	05	13	06	09	20	08	07	

Table 7.TMTP$(TMP3)^{2(7)}(T_u = \infty)$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	⑫			③					15
	00	M	M	00	M	M	08	M	
S_2						⑩			10
	M	00	M	03	00	00	M	00	
S_3			⑫						12
	M	M	00	M	08	M	00	M	
S_4			①			⑤			06
	00	05	00	00	M	00	M	M	
S_5				⑤		⑤		⑦	17
	M	M	M	M	00	14	M	00	
S_6		⑤		③	④		⑧		20
	11	00	M	05	00	M	00	M	
b_j	12	05	13	06	09	20	08	07	

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Table 8. TMTP $(TMP3)^{3(1)}(T_u = \infty)$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	$\underline{00}$	M	M	$\underline{00}$	M	M	08	M	15
S_2	M	$\underline{00}$	M	03	$\underline{00}$	$\underline{00}$	M	$\underline{00}$	10
S_3	M	M	M	M	08	M	$\underline{00}$	M	12
S_4	$\underline{00}$	05	$\underline{00}$	$\underline{00}$	M	$\underline{00}$	M	M	06
S_5	M	M	M	M	$\underline{00}$	14	M	$\underline{00}$	17
S_6	11	$\underline{00}$	M	05	$\underline{00}$	M	$\underline{00}$	M	20
b_j	12	05	13	06	09	20	08	07	

Step 2. $T1^{prof}(1) = 35 > T1^{min}$, therefore go to Step 3.

Step 3. Set $T1^{prof}(1) = 35 = T_u$. For finding the minimum shipment time for Phase-3 with respect to Phase-1 under the restriction that Phase-1 time is strictly less than T_u, solve the problem $(TMP3)^0(T_u = 35)$ (Table 9) with time entries as

$$t'_{ij} = \left\{ \begin{array}{ll} M & \text{if } t_{ij} \geq 35 \text{ for } (i,j) \in P_1 \\ t_{ij} & (i,j) \in P_3 \\ 0 & \text{otherwise} \end{array} \right\}$$

Table 9. TMTP $(TMP3)^0$ $(T_u = 35)$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	⑩ $\underline{00}$	⑤ 00	00	$\underline{00}$	00	32	08	42	15
S_2	40	$\underline{00}$	29	03	$\underline{00}$	② 00	⑧ 00	$\underline{00}$	10
S_3	① 00	21	00	⑥ 00	08	⑤ 00	$\underline{00}$	24	12
S_4	① $\underline{00}$	05	⑤ $\underline{00}$	$\underline{00}$	00	00	37	00	06
S_5	29	00	⑧ 00	19	⑨ $\underline{00}$	14	00	00	17
S_6	11	$\underline{M}$	42	05	$\underline{00}$	⑬ 00	$\underline{M}$	⑦ 00	20
b_j	12	05	13	06	09	20	08	07	

From an optimal feasible solution of $(TMP3)^0(T_u = 35)$, the first triad of this sequence is obtained as $(T1^0(T_u), T2^0(T_u), T3^0(T_u)) = (17, 32, 00)$.

Proceeding on the lines of algorithm from Step 3 to Step 7 repeatedly for every generated triad

Table 10. All proficient triads with respect to Phase-1

Value of l	Triad recorded	$T1^{prof}(1) + T2^{prof}(1) + T3^{prof}(1)$
1	(35,02,14)	51
2	(27,02,14)	43
3	(17,32,00)	49
4	(09,32,00)	41
5	(08,32,00)	40
6	(06,32,00)	38
7	(04,27,14)	45
8=v (last)	(00,27,14)	41

$(T1^{k}(T_u), T2^{k}(T_u), T3^{k}(T_u))$, a sequence of triads is obtained as follows.

$$(T1^{0}(T_u), T2^{0}(T_u), T3^{0}(T_u)) = (17, 32, 00)$$
$$(T1^{1}(T_u), T2^{1}(T_u), T3^{1}(T_u)) = (27, 17, 11)$$
$$(T1^{2}(T_u), T2^{2}(T_u), T3^{2}(T_u)) = (27, 15, 11)$$
$$(T1^{3}(T_u), T2^{3}(T_u), T3^{3}(T_u)) = (27, 3, 14)$$
$$(T1^{4}(T_u), T2^{4}(T_u), T3^{4}(T_u)) = (27, 02, 14) = (T1^{prof}(2), T2^{prof}(2), T3^{prof}(2))$$
$$= \text{proficient triad with respect to Phase} - 1$$

Repeating Step 2 to Step 7 for $l = 2$, the next proficient triad with respect to Phase-1 under the restriction that Phase-1 time is strictly less than $T1^{prof}(2) = 27$ is obtained as $(17, 32, 00) = (T1^{prof}(3), T2^{prof}(3), T3^{prof}(3))$.

Continuing so on, the recorded proficient triads with respect to Phase-1 are as given in Table 10.

Step 13. The optimal value of 3-PhTP(ex1) is

$$\min_{\{l=1,2,...,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right] = 38$$

The corresponding triad $(T1^{prof}(6), T2^{prof}(6), T3^{prof}(6)) = (06, 32, 00)$ is an optimal triad for 3-PhTP(ex1).

Solving 3-PhTP(ex1) by generating all proficient triads with respect to Phase-2:

In order to find proficient triads with respect to Phase-2, start with the triad corresponding to Phase-3 minimum time, i.e (35,32,00) as in above case, but search for the triads having minimum sum of Phase-1 and Phase-3 times corresponding to Phase-2 time.

For obtaining first proficient triad with respect to Phase-2 from the triad (35,32,00), a sequence of intermediate triads is obtained such that Phase-1 time decreases and corresponding minimum time of Phase-3 increases in such a way that the sum of Phase-1 and Phase-3 times decreases strictly. The last triad of such a sequence is the required first proficient triad with respect to Phase-2.

Proceeding on these lines, the first proficient triad with respect to Phase-2 is obtained as (06,32,00). The sequence of intermediate triads generated in obtaining first proficient triad (06,32,00) is as follows.

(35,32,00); (17,32,00); (09,32,00); (08,32,00); (06,32,00)=$(T1^{prof}(1), T2^{prof}(1), T3^{prof}(1))$

Next, to find proficient triad with respect to Phase-2 under the restriction that Phase-2 time

Table 11. All intermediate and proficient triads with respect to Phase-2 for 3-PhTP(ex1)

Value of l	sequence of inter-mediate triads	l-th proficient triad $(T1^{prof}(l), T2^{prof}(l), T3^{prof}(l))$	$T1^{prof}(1) + T2^{prof}(1) + T3^{prof}(1)$
1	(35,32,00); (17,32,00); (09,32,00); (08,32,00); (06,32,00)	(06,32,00)	38
2	(35,27,00); (27,27,05); (17,27,11); (09,27,14); (08,27,14); (06,27,14); (04,27,14); (00,27,14)	(00,27,14)	41
3	(35,17,00)	(35,17,00)	52
4	(35,15,11); (27,15,11)	(27,15,11)	53
5	(35,12,14); (27,12,14)	(27,12,14)	53
6	(27,03,14)	(27,03,14)	44
7	(35,02,14); (27,02,14)	(27,02,14)	43
8=v (last)	(35,00,29); (27,00,29); (13,00,42); (00,00,42)	(00,00,42)	42

is strictly less than $T2^{prof}(1) = 32$. Table 11 records all proficient triads with respect to Phase-2 and intermediate triads generated while obtaining proficient triads.

Clearly, the optimal value of 3-PhTP(ex1) is

$$\min_{\{l=1,2,...,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right] = 38$$

and the corresponding triad $(T1^{prof}(1), T2^{prof}(1), T3^{prof}(1)) = (06, 32, 00)$ is an optimal triad for 3-PhTP(ex1), which is same as obtained in earlier case where proficient triads with respect to Phase-1 are generated.

Solving 3-PhTP(ex1) by generating all proficient triads with respect to Phase-3:

In order to find proficient triads with respect to Phase-3, start with the triad corresponding to Phase-1 or Phase-2 minimum time and search for the triads having minimum sum of Phase-1 and Phase-2 times corresponding to Phase-3 time.

Table 12. All intermediate and proficient triads with respect to Phase-3 for 3-PhTP(ex1)

Value of l	sequence of intermediate triads	l-th proficient triad $(T1^{prof}(l), T2^{prof}(l), T3^{prof}(l))$	$T1^{prof}(1) + T2^{prof}(1) + T3^{prof}(1)$
1	(35,00,42); (13,00,42); (04,00,42); (00,00,42)	(00,00,42)	42
2	(35,00,32); (27,00,32)	(27,00,32)	59
3	(35,00,29); (27,00,29)	(27,00,29)	56
4	(35,00,24); (27,02,21); (00,27,24)	(00,27,24)	51
5	(35,02,21); (27,02,21); (00,27,14)	(00,27,14)	41
6	(35,15,11); (27,15,11); (09,32,11); (08,32,11); (06,32,11)	(06,32,11)	49
7	(35,17,05); (17,32,08); (15,32,08); (13,32,08); (09,32,08); (08,32,08); (06,32,08)	(06,32,08)	46
8	(35,17,05); (17,32,05); (15,32,05); (13,32,05); (09,32,05); (08,32,05); (06,32,05)	(06,32,05)	43
9	(35,17,00); (17,32,03); (15,32,03); (13,32,03); (09,32,03); (08,32,00); (06,32,03)	(06,32,03)	41
10=v (last)	(35,17,00); (17,32,00); (13,32,00); (08,32,00); (06,32,00)	(06,32,00)	38

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For 3-PhTP(ex1), start with the triad $(35, 00, 42)$ corresponding to Phase-2 minimum time obtained by solving TMTP $(TMP2)^0$ and obtain first proficient triad with respect to Phase-3. A sequence of intermediate triads is obtained, starting from $(35, 00, 42)$ such that Phase-1 time decreases and corresponding minimum time of Phase-2 increases in such a way that the sum of Phase-1 and Phase-2 times also decreases strictly. The last triad of such a sequence is the required first proficient triad with respect to Phase-3.

Proceeding on these lines, the first proficient triad with respect to Phase-3 is obtained as $(00, 00, 42) = (T1^{prof}(1), T2^{prof}(1), T3^{prof}(1))$.

Next, to find proficient triad with respect to Phase-3 under the restriction that Phase-3 time is strictly less than $T3^{prof}(1) = 42$. Table 12 records all proficient triads with respect to Phase-3 and all intermediate triads generated while obtaining proficient triads.

Clearly, the optimal value of 3-PhTP(ex1) is

$$\min_{\{l=1,2,\ldots,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right] = 38$$

and the corresponding triad $(T1^{prof}(10), T2^{prof}(10), T3^{prof}(10)) = (06, 32, 00)$ is an optimal triad for 3-PhTP(ex1), which is same as obtained in two cases before.

Example 2. Consider a larger 3-PhTP (namely 3-PhTP(ex2) as in Table 13) with number of origins as $m = 10$ and number of terminals as $n = 10$. Here, cells with underlined entries denote Phase-1 links, cells with encircled entries denote Phase-2 links and remaining cells correspond to Phase-3 links.

Table 13. 3-PhTP(ex2)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	a_i
S_1	(21)	10	48	19	(31)	58	43	(39)	60	29	05
S_2	29	43	(21)	09	35	(12)	58	28	10	(45)	12
S_3	60	(45)	17	(12)	58	19	43	17	(45)	10	20
S_4	35	(12)	60	29	28	(31)	10	09	(39)	48	14
S_5	(45)	35	09	(61)	19	29	(31)	58	17	(21)	18
S_6	10	(39)	29	(39)	43	(61)	28	60	(12)	19	21
S_7	09	48	(31)	12	10	28	(45)	58	43	35	07
S_8	(12)	28	58	35	(21)	09	60	19	29	(61)	25
S_9	19	48	28	(21)	43	10	09	(45)	43	17	15
S_{10}	(31)	17	35	48	(12)	60	29	(21)	09	(39)	06
b_j	18	07	14	06	21	12	20	12	06	27	

The distinct time entries in three phases arranged in increasing order are as follows.
Phase-1: $t_1^1 (= 10) < t_1^2 (= 17) < t_1^3 (= 29) < t_1^4 (= 35) < t_1^5 (= 48) < t_1^6 (= 58) = t_1^\alpha; \alpha = 06$

Phase-2: $t_2^1 (= 12) < t_2^2 (= 21) < t_2^3 (= 31) < t_2^4 (= 39) < t_2^5 (= 45) < t_2^6 (= 61) = t_2^\beta; \beta = 06$

Phase-3: $t_3^1 (= 09) < t_3^2 (= 19) < t_3^3 (= 28) < t_3^4 (= 43) < t_3^5 (= 60); \gamma = 05$

On solving the standard TMTPs $(TMP1)^0$, $(TMP2)^0$ and $(TMP3)^0$, the minimum shipment times of Phase-1. Phase-2 and Phase-3 are obtained as follows.

$$T1^{min} = 0, \ T2^{min} = 0 \text{ and } T3^{min} = 0$$

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Proceeding on the lines of algorithm as in Example 1, all intermediate and proficient triads with respect to Phase-1 are recorded in Table 14.

The optimal value of 3-PhTP(ex2) is

$$\min_{\{l=1,2,\ldots,v\}} \left[T1^{prof}(l) + T2^{prof}(l) + T3^{prof}(l) \right] = 40$$

and the corresponding triad $(T1^{prof}(6), T2^{prof}(6), T3^{prof}(6)) = (10, 21, 09)$ is an optimal triad for 3-PhTP(ex2).

Table 14. All intermediate and proficient triads with respect to Phase-1 for 3-PhTP(ex2)

Value of l	sequence of intermediate triads	l-th proficient triad $(T1^{prof}(l), T2^{prof}(l), T3^{prof}(l))$	$T1^{prof}(1) + T2^{prof}(1) + T3^{prof}(1)$
1	(58,61,00); (58,45,00); (58,39,00); (58,31,00); (58,21,00); (58,00,09)	(58,00,09)	67
2	(35,61,00); (35,45,00); (48,39,00); (48,21,00); (48,00,19)	(48,00,19)	67
3	(35,61,00); (35,45,00); (29,39,00); (35,31,00); (35,21,00); (35,00,19)	(35,00,19)	54
4	(29,61,00); (29,45,00); (29,39,00); (29,31,00); (29,21,09); (29,00,28); (29,00,19)	(29,00,19)	48
5	(17,61,00); (17,45,00); (17,39,00); (17,21,09)	(17,21,09)	47
6	(10,61,00); (10,45,00); (10,31,09); (10,21,09)	(10,21,09)	40
7=v (last)	(00,61,19); (00,45,19)	(00,45,19)	64