Measuring Granger Causality in Quantiles: Supplementary Material

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This document provides the necessary assumptions and the proofs of the theoretical results in the main text, and some additional simulation results. Appendix A of this document reports a set of standard assumptions that have been widely used in the literature on nonparametric estimation and inference; see for example Kong et al. (2010) and Noh et al. (2013) among others. Appendix B contains the detailed proofs of the theoretical results developed in sections 5, 6 and 7 of the main text. In particular, it contains the proofs of theorems 1, 2, 3, 4, and 5 and propositions 3 and 4 of the main text. Appendix C provides four auxiliary lemmas which are useful to prove the results in sections 5-7 of the main text. Finally, Appendix D contains the tables of some additional simulation results that examine a bootstrap bias-corrected estimator of measure of Granger causality in quantiles.

Appendix A: Assumptions

First of all, let $\{(X_t, Y_t)\}$ be a jointly stationary process. Since we are interested in time series data, we need to specify the dependence in the processes of interest. In what follows, we define the mixing dependence that we consider in this paper. The stationary stochastic process $\{(X_t, Y_t)\}$ is strongly mixing, with $\gamma(k)$ its strong mixing coefficient, if

$$\gamma(k) = \sup_{A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{k}^{\infty}} \left| \mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B) \right| \to 0 \text{ as } k \to \infty,$$

with $\mathcal{F}_a^b = \sigma\left(\{(X_t, Y_t)\}_{t=a}^{b}\right)$, where $\sigma(\cdot)$ means the smallest sigma algebra. Furthermore, let V_x and V_z be two open convex sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_1+d_2}$, respectively. We now consider the following assumptions:

A.1. The processes $\{(X_t, Y_t)\}$ are strongly mixing with mixing coefficients $\gamma(k)$ satisfying

$$\sum_{k=1}^{\infty} k^{\alpha} \left[\gamma(k)\right]^{1-2/\nu_2} < \infty,$$

for some $\nu_2 > 2$ and $\alpha > \max\{(p + d_1 + 1)(1 - 2/\nu_2)/d_1, (q + d_1 + d_2 + 1)(1 - 2/\nu_2)/(d_1 + d_2)\}$.

A.2. All partial derivatives of $\bar{q}_\tau(\underline{x})$ up to order $p+1$ exist and are continuous for all $\underline{x} \in V_x$, and there exists a constant $C_1 > 0$ such that $|D^{\underline{r}} \bar{q}_\tau(\underline{x})| \leq C_1$, for all $\underline{x} \in V_x$ and $|\underline{r}| = p+1$. All partial derivatives of $q_\tau(\underline{z})$ up to order $q+1$ exist and are continuous for all $\underline{z} \in V_z$, and there exists a constant $C_2 > 0$ such that $|D^{\underline{r}} q_\tau(\underline{z})| \leq C_2$, for all $\underline{z} \in V_z$ and $|\underline{r}| = q+1$.

A.3. The marginal density of $\varepsilon_t = X_t - q_\tau(\underline{Z}_{t-1})$ is bounded and satisfies $E(\varphi(\varepsilon_t)|\underline{Z}_{t-1}) = 0$.

A.4. For all e in a neighbourhood of zero, the conditional density $f_{\varepsilon|\underline{Z}_{-1}}(e|\underline{z})$ of $\varepsilon_t = X_t - q_\tau(\underline{Z}_{t-1})$ given $\underline{Z}_{t-1} = \underline{z}$ satisfies

$$\left| f_{\varepsilon|\underline{Z}_{-1}}(e|\underline{z}_1) - f_{\varepsilon|\underline{Z}_{-1}}(e|\underline{z}_2) \right| \leq K_e \|\underline{z}_1 - \underline{z}_2\|,$$

where K_e is a positive constant depending on e. Further, the conditional density is positive for $e = 0$ for all values of $\underline{z} \in V_z$, and its first partial derivative with respect to e, $D^1 f_{\varepsilon|\underline{Z}_{-1}}(e|\underline{z})$, is bounded for all $\underline{z} \in V_z$ and e in a neighbourhood of zero.

A.5. The weight function $w(\underline{z})$ is continuous, and its support $\mathcal{D} \subset V_z$ is compact and has non-empty interior.

A.6. The kernel function $K(\cdot)$ has a compact support and $\left| H_{\underline{j}}(\underline{u}) - H_{\underline{j}}(\underline{v}) \right| \leq \|\underline{u} - \underline{v}\|$ for all j with $0 \leq \underline{j} \leq \max\{2p+1, 2q+1\}$, where $H_{\underline{j}}(\underline{u}) = \underline{u}^{\underline{j}} K(\underline{u})$.

A.7. The probability density function of $\underline{Z}_{t-1}$, $f_{\underline{Z}}(\underline{z})$, is positive and bounded with bounded first-order derivatives on V_z. The joint probability density of $(\underline{Z}_0, \underline{Z}_l)$ satisfies $f_{(\underline{Z}_0, \underline{Z}_l)}(\underline{u}, \underline{v}; l) \leq C < \infty$ for all $l \geq 1$.

A.8. The conditional density $f_{\underline{Z}_{-1}|X}$ of $\underline{Z}_{t-1}$ given X_t exists and is bounded. The conditional density function $f_{(\underline{Z}_0, \underline{Z}_l)|(X_1, X_l+1)}$ of $(\underline{Z}_0, \underline{Z}_l)$ given $(X_1, X_l + 1)$ exists and is bounded for all $l \geq 1$.

A.9. The bandwidth sequences h_1 and h_2 satisfy $h_1 \to 0$, $Th_1^{d_1+2(p+1)}/\log T = O(1)$, $h_2 \to 0$, and $Th_2^{d_1+d_2+2(q+1)}/\log T = O(1)$ as $T \to \infty$. Furthermore, we assume $Th_2^{2(d_1+d_2)}/(\log T)^3 \to \infty$, $h_1 = o(h_2)$, and $h_2^{d_1+d_2} = o(h_1^{d_1})$.

A.10. The bootstrap bandwidth h^* satisfies $h^* \to 0$ and $Th^{*\ d_1+d_2+2(q+1)}/(\log T)^\lambda = O(1)$, for some $\lambda > 0$ as $T \to \infty$.

The assumptions presented here are frequently seen for nonparametric smoothing in multivariate time series analysis, see Masry (1996) and Kong et al. (2010). Assumptions A.1-A.2, A.6-A.8 and A.9 are standard. Assumptions A.4 and A.5 are required to derive the Bahadur representations in Lemmas 3-4 in Appendix C. Assumption A.10 is assumed to guarantee the consistency of the smoothed local bootstrap.

Appendix B: Proofs of the main results

This section provides the proofs of the main theoretical results developed in sections 5, 6 and 7 of the main text.

Proof of Theorem 1: Theorem 1 can be proved by combing the first order Taylor expansion of $C_\tau\,\widehat{(Y \to X)}$ around 1 (i.e. using $\ln y \approx y-1$) and the asymptotic Bahadur representations in Lemmas 3 and 4 of Appendix C and with the equality $\hat{a}/\hat{b} = a/b + \hat{b}^{-1}[(\hat{a}-a) - (\hat{b}-b)(a/b)]$. $\square$

Proof of Theorem 2: Note that for any x, y,

$$\rho_\tau(x-y) - \rho_\tau(x) = (-y)\varphi(x) + 2(y-x)[1(y > x > 0) - 1(y < x < 0)].$$

Let $\widehat{\bar{\bar{d}}}(\underline{x}) = \widehat{\bar{\bar{q}}}_\tau(\underline{x}) - \bar{q}_\tau(\underline{x})$ and $\widehat{d}(\underline{z}) = \hat{q}_\tau(\underline{z}) - q_\tau(\underline{z})$. By straightforward calculation, under the null hypothesis of no causality, we obtain

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} \rho_\tau(X_t - \widehat{\bar{\bar{q}}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - \frac{1}{T}\sum_{t=1}^{T} \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1}) \\
=&\frac{1}{T}\sum_{t=1}^{T}\left[\left(\hat{q}_\tau(\underline{Z}_{t-1}) - q_\tau(\underline{Z}_{t-1})\right) - \left(\widehat{\bar{\bar{q}}}_\tau(\underline{X}_{t-1}) - \bar{q}_\tau(\underline{X}_{t-1})\right)\right] w(\underline{Z}_{t-1})\varphi(\varepsilon_t) \\
&+\frac{2}{T}\sum_{t=1}^{T}\left(X_t - \hat{q}_\tau(\underline{Z}_{t-1})\right)\left\{1(\widehat{d}(\underline{Z}_{t-1}) > \varepsilon_{1t} > 0) - 1(\widehat{d}(\underline{Z}_{t-1}) < \varepsilon_t < 0)\right\} w(\underline{Z}_{t-1}) \\
&-\frac{2}{T}\sum_{t=1}^{T}\left(X_t - \widehat{\bar{\bar{q}}}_\tau(\underline{X}_{t-1})\right)\left\{1(\widehat{\bar{\bar{d}}}(\underline{X}_{t-1}) > \varepsilon_t > 0) - 1(\widehat{\bar{\bar{d}}}(\underline{X}_{t-1}) < \varepsilon_t < 0)\right\} w(\underline{Z}_{t-1}) \\
:=&A_T + B_T + C_T.
\end{aligned}$$

From the above decomposition, we will show that under the assumed assumptions, the term A_T is asymptotically normal, and the terms B_T and C_T are asymptotically negligible.

Now, let us first show the asymptotic negligibility of term B_T. Define $I(w) = \{t : \underline{Z}_{t-1} \in \mathcal{D}, t = 1, \ldots, T\}$. Note that $X_t - \hat{q}_\tau\left(\underline{Z}_{t-1}\right) = -\widehat{d}\left(\underline{Z}_{t-1}\right) + \varepsilon_t$. Then,

$$\begin{aligned}
|B_T| &\le \frac{2}{T}\sum_{t=1}^{T} w(\underline{Z}_{t-1})\left|X_t - \hat{q}_\tau(\underline{Z}_{t-1})\right| 1\left(|\varepsilon_t| < \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right|\right) \\
&\le \frac{2}{T}\sum_{t=1}^{T} w(\underline{Z}_{t-1})\left(\left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| + |\varepsilon_t|\right) 1\left(|\varepsilon_t| < \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right|\right) \\
&\le \frac{4}{T}\sum_{t=1}^{T} w(\underline{Z}_{t-1})\left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| 1\left(|\varepsilon_t| < \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right|\right) \\
&\le 4\max_{t\in I(w)}\left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \max_{\underline{z}\in\mathcal{D}} w(\underline{z})\,\frac{1}{T}\sum_{t=1}^{T} 1\left(|\varepsilon_t| < \max_{s\in I(w)}\left|\widehat{d}\left(\underline{Z}_{s-1}\right)\right|\right).
\end{aligned}$$

From the Glivenko-Cantelli Theorem for strictly stationary sequences, we have

$$\sup_{a\in\mathbb{R}}\left|\frac{1}{T}\sum_{t=1}^{T} 1\left(|\varepsilon_t| < a\right) - Pr\left(|\varepsilon| < a\right)\right| = O_p\left(T^{-1/2}\right),$$

It thus follows that

$$
\begin{aligned}
|B_T| \leq & 4 \max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \max_{\underline{z}\in\mathcal{D}} w(\underline{z}) \left\{ Pr\left(|\varepsilon| < \max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \right) + O_p\left(T^{-1/2}\right) \right\} \\
= & 4 \max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \max_{\underline{z}\in\mathcal{D}} w(\underline{z}) \left\{ F_\varepsilon\left(\max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \right) - F_\varepsilon\left(- \max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \right) \right\} \\
& + 4 \max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \max_{\underline{z}\in\mathcal{D}} w(\underline{z})\, O_p\left(T^{-1/2}\right) \\
\leq & C\left(\max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| \right)^2 + CT^{-1/2} \max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right|,
\end{aligned}
$$

where the third step follows from the Taylor expansion of F_ε, bounded marginal density of ε_t in Assumption A.3, and bounded weight function $w(\cdot)$ in Assumption A.5. From Kong et al. (2010), we have

$$
\max_{t\in I(w)} \left|\widehat{d}\left(\underline{Z}_{t-1}\right)\right| = O_p\left(\frac{\log T}{Th_2^d}\right)^{3/4},
$$

It follows that $B_T = O_p\left(\left(\frac{\log T}{Th_2^d}\right)^{3/2} + T^{-1/2}\left(\frac{\log T}{Th_2^d}\right)^{3/4}\right) = o_p\left(\left(Th_2^{d/2}\right)^{-1}\right)$ under Assumption A.9.

Similar to the term B_T, it can be proved that the term $C_T = o_p\left(\left(Th_1^{d_1/2}\right)^{-1}\right) = o_p\left(\left(Th_2^{d/2}\right)^{-1}\right)$ under $h_2^d = o(h_1^{d_1})$ in Assumption A.9. It follows that it is sufficient to establish that $Th_2^{d/2}A_T$ converges in distribution to a normal random variable with asymptotic variance given by $\tilde{\sigma}_{0\tau}^2 := \kappa\left(\tau\right)^2 \sigma_{0\tau}^2$, for $\kappa\left(\tau\right) = E\left[\rho_\tau\left(X_t - q_\tau\left(\underline{Z}_{t-1}\right)\right) w\left(\underline{Z}_{t-1}\right)\right]$.

Using Lemmas 1 and 2 of Appendix C, we have

$$
\begin{aligned}
A_T = & -\frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s=1,s\neq t}^{T} w(\underline{Z}_{t-1}) \underline{e}_1' \frac{H_T^{-1}}{h_2^d} S_{T,q}^{-1}(\underline{Z}_{t-1}) K_{h_2}(\underline{Z}_{s-1} - \underline{Z}_{t-1}) \mu(\underline{Z}_{s-1} - \underline{Z}_{t-1}) \varphi(\varepsilon_t)\varphi(\varepsilon_s) \\
& + \frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s=1,s\neq t}^{T} w(\underline{Z}_{t-1}) \overline{e}_1' \frac{H_T^{-1}}{h_1^{d_1}} S_{T,p}^{-1}(\underline{X}_{t-1}) K_{h_1}(\underline{X}_{s-1} - \underline{X}_{t-1}) \mu(\underline{X}_{s-1} - \underline{X}_{t-1}) \varphi(\varepsilon_t)\varphi(\varepsilon_s) \\
& + o_p\left(\left(Th_2^{d/2}\right)^{-1}\right) \\
:= & A_{1T} + A_{2T} + o_p\left(\left(Th_2^{d/2}\right)^{-1}\right),
\end{aligned}
$$

where the negligible terms with $t = s$ have been dropped to apply U-statistic theory due to the leave one observation out in the estimation part. We will show that $Th_2^{d/2}A_{1T}$ converges in distribution and $A_{2T} = o_p\left(\left(Th_2^{d/2}\right)^{-1}\right)$ under our assumptions. First of all, to facilitate our analysis, from the notion of "equivalent kernel" representation for local polynomial estimator [see Fan and Gijbels, 1996, pp.63-64], we get

$$
\begin{aligned}
A_{1T} = & \frac{1}{T(T-1)} \sum_{t=1}^{T} \sum_{s=1,s\neq t}^{T} \frac{w(\underline{Z}_{t-1})}{f_{\varepsilon|\underline{Z}}(0|\underline{Z}_{t-1}) f_{\underline{Z}}(\underline{Z}_{t-1})} K_{h_2}(\underline{Z}_{t-1} - \underline{Z}_{s-1}) \varphi(\varepsilon_t)\varphi(\varepsilon_s) + o_p\left(\left(Th_2^{d/2}\right)^{-1}\right) \\
\equiv & \overline{A}_{1T} + o_p\left(\left(Th_2^{d/2}\right)^{-1}\right), \quad \text{say.}
\end{aligned}
$$

Note that we can rewrite $Th_2^{d/2}\overline{A}_{1T}$ into a standard U-statistic form with a symmetrized kernel depending on the sample size T, i.e.

$$Th_2^{d/2}\overline{A}_{1T} = \frac{2}{T-1}\sum_{1\leq t<s\leq T} U_T(\chi_t,\chi_s), \tag{1}$$

where $\chi_t = (\underline{Z}_{t-1},\varepsilon_t)$, $U_T(\chi_t,\chi_s) = \eta_T(\chi_t,\chi_s) + \eta_T(\chi_s,\chi_t)$, and

$$\eta_T(\chi_t,\chi_s) = \frac{w(\underline{Z}_{t-1})}{2f_{\varepsilon|\underline{Z}}(0|\underline{Z}_{t-1})f_{\underline{Z}}(\underline{Z}_{t-1})}\frac{1}{h_2^{\frac{d}{2}}}K\left(\frac{\underline{Z}_{t-1}-\underline{Z}_{s-1}}{h_2}\right)\varphi(\varepsilon_t)\varphi(\varepsilon_s).$$

Note that $E[U_T(\chi_t,\chi_s)] = E[\eta_T(\chi_t,\chi_s)] = E[U_T(\chi_t,\chi_s)|\chi_t] = E[\eta_T(\chi_t,\chi_s)|\chi_t] = 0$ under Assumption A.3. So the previous U-statistic is a degenerate second order U-statistic. We can apply a central limit theorem (CLT) for second order degenerate U-statistic with strong mixing processes. Under Assumptions A.1, A.3, A.6, and A.9, one can verify that the conditions of Theorem A.1 in Gao (2007) are satisfied for kernel $U_T(\chi_t,\chi_s)$ so that a CLT applies to the term $Th_2^{d/2}\overline{A}_{1T}$. Its asymptotic variance is given by

$$\begin{aligned}\tilde{\sigma}_{0\tau}^2 = \lim_{T\to\infty} 2E_tE_s\left[U_T^2(\chi_t,\chi_s)\right] &= \lim_{T\to\infty} 2E_tE_s\left[\eta_T(\chi_t,\chi_s)^2 + \eta_T(\chi_s,\chi_t)^2 + 2\eta_T(\chi_t,\chi_s)\eta_T(\chi_s,\chi_t)\right] \\ &= 2\tau^2(1-\tau)^2\int K^2(\underline{u})\,d\underline{u}\int\frac{w^2(\underline{z})}{f_{\varepsilon|\underline{Z}}^2(0|\underline{z})}\,d\underline{z} \\ &:= \kappa(\tau)^2\sigma_{0\tau}^2,\end{aligned}$$

where E_t denotes the expectation with respect to χ_t. For example, by straightforward calculation of conditional expectation, we have

$$\begin{aligned}&\lim_{T\to\infty} E_tE_s\left[\eta_T(\chi_t,\chi_s)^2\right] \\ =&\frac{1}{4}\tau^2(1-\tau)^2\lim_{T\to\infty}\int\int\frac{w^2(\underline{z}_1)}{f_{\varepsilon|\underline{Z}}^2(0|\underline{z}_1)f_{\underline{Z}}^2(\underline{z}_1)}\frac{1}{h_2^d}K^2\left(\frac{\underline{z}_1-\underline{z}_2}{h_2}\right)f_{\underline{Z}}(\underline{z}_1)f_{\underline{Z}}(\underline{z}_2)\,d\underline{z}_1\,d\underline{z}_2 \\ =&\frac{1}{4}\tau^2(1-\tau)^2\int K^2(\underline{u})\,d\underline{u}\int\frac{w^2(\underline{z})}{f_{\varepsilon|\underline{Z}}^2(0|\underline{z})}\,d\underline{z}\end{aligned}$$

by standard use of change of variables and Assumptions A.7 and A.9. The U-statistic representation in (1), together with the form of asymptotic variance $\tilde{\sigma}_{0\tau}^2$, implies that $Th_2^{d/2}A_{1T} = Th_2^{d/2}\overline{A}_{1T} + o_p(1) \xrightarrow{d} \mathcal{N}\left(0,\tilde{\sigma}_{0\tau}^2\right)$.

Observing that by using almost the same steps as in proving the asymptotic normality of $Th_2^{d/2}A_{1T}$, we can prove that $Th_1^{d_1/2}A_{2T}$ converges in distribution to a normal variable, and therefore $Th_1^{d_1/2}A_{2T} = O_p(1)$. Thus, $Th_2^{d/2}A_T = Th_2^{d/2}A_{1T} + \left(h_2^{d/2}/h_1^{d_1/2}\right)\left(Th_1^{d_1/2}A_{2T}\right) + o_p(1) \xrightarrow{d} \mathcal{N}\left(0,\tilde{\sigma}_{0\tau}^2\right)$ by the assumption $h_2^d = o\left(h_1^{d_1}\right)$ in A.9.

In addition, a consistent estimator for $\tilde{\sigma}_{0\tau}^2$ is given by

$$\begin{aligned}\widehat{\tilde{\sigma}}_{0\tau}^2 =&2\tau^2(1-\tau)^2\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T}\frac{w^2(\underline{Z}_{t-1})}{\widehat{f}_{\varepsilon,\underline{Z}}^2(0,\underline{Z}_{t-1})}\frac{1}{h_2^d}K^2\left(\frac{\underline{Z}_{t-1}-\underline{Z}_{s-1}}{h_2}\right) \\ =&2\tau^2(1-\tau)^2\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T}\frac{w^2(\underline{Z}_{t-1})}{f_{\varepsilon,\underline{Z}}^2(0,\underline{Z}_{t-1})}\frac{1}{h_2^d}K^2\left(\frac{\underline{Z}_{t-1}-\underline{Z}_{s-1}}{h_2}\right) + o_p(1),\end{aligned}$$

where $\widehat{f}_{\varepsilon,\underline{Z}}(0,\underline{Z}_{t-1})$ is the leave-one-out kernel density estimator defined in the main text for $f_{\varepsilon,\underline{Z}}(0,\underline{Z}_{t-1}) \equiv f_{\varepsilon|\underline{Z}}(0|\underline{Z}_{t-1})f_{\underline{Z}}(\underline{Z}_{t-1})$. As a consequence, the main term of $\widehat{\widetilde{\sigma}}^2_{0\tau}$ can also be written into a standard U-statistic form with a symmetrized kernel

$$H_T(\underline{Z}_{t-1},\underline{Z}_{s-1}) = 2\tau^2(1-\tau)^2\left(\frac{w^2(\underline{Z}_{t-1})}{f^2_{\varepsilon,\underline{Z}}(0,\underline{Z}_{t-1})} + \frac{w^2(\underline{Z}_{s-1})}{f^2_{\varepsilon,\underline{Z}}(0,\underline{Z}_{s-1})}\right)\frac{1}{h_2^d}K^2\left(\frac{\underline{Z}_{t-1}-\underline{Z}_{s-1}}{h_2}\right).$$

Note that in contrast to (1), $\widehat{\widetilde{\sigma}}^2_{0\tau}$ is a non-degenerate second order U-statistic and by the usual Hoeffding decomposition, one can thus show that $\widehat{\widetilde{\sigma}}^2_{0\tau} = \widetilde{\sigma}^2_{0\tau} + o_p(1)$.

Finally, observing that by Taylor expansion of $\ln y$ around 1 (i.e. $\ln y \approx y-1$) and using the asymptotic equivalence of $\widehat{\kappa(\tau)}$ to $\kappa(\tau) = E[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})]$ stated in Lemma 4 of Appendix C, together with Slutsky's theorem, we have

$$\begin{aligned}
Th_2^{d/2}C_\tau(\widehat{Y \to X}) &= Th_2^{d/2}\left(\frac{T^{-1}\sum_{t=1}^T \rho_\tau(X_t - \widehat{\bar{q}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})}{T^{-1}\sum_{t=1}^T \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})} - 1\right) + o_p(1) \\
&= \kappa(\tau)^{-1} Th_2^{d/2}A_T + o_p(1) \\
&\xrightarrow{d} \mathcal{N}\left(0,\sigma^2_{0\tau}\right),
\end{aligned}$$

where $\sigma^2_{0\tau} := \widetilde{\sigma}^2_{0\tau}/\kappa(\tau)^2$. It is straightforward to show that $\widehat{\sigma}^2_{0\tau} = \widehat{\widetilde{\sigma}}^2_{0\tau}/\widehat{\kappa(\tau)}^2$ is a consistent estimator for $\sigma^2_{0\tau}$. Thus, our test statistic $\widehat{\Gamma}_\tau = Th_2^{d/2}C_\tau(\widehat{Y \to X})/\widehat{\sigma}_{0\tau} \xrightarrow{d} \mathcal{N}(0,1)$. This ends the proof of Theorem 2. □

Proof of Proposition 3: This result can be shown by following the same steps as in the proof of Theorem 2. Noting that, under the fixed alternative hypothesis H_1 in Equation (15) of the main text, we have

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^T \rho_\tau(X_t - \widehat{\bar{q}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - \frac{1}{T}\sum_{t=1}^T \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1}) \\
=&\left(\frac{1}{T}\sum_{t=1}^T \rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - \frac{1}{T}\sum_{t=1}^T \rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right) \\
&+ A_T + B_T + C_T \\
:=&D_T + A_T + B_T + C_T,
\end{aligned}$$

where the last three terms A_T, B_T, and C_T are as defined before in the proof of Theorem 2. Following the same arguments as those in Theorem 2, $Th_2^{d/2}(A_T + B_T + C_T) = O_p(1)$. As a matter of fact, one can furthermore prove that all A_T, B_T and C_T are of order $o_p\left(T^{-1/2}\right)$, see the proof of lemma 3 in Noh et al. (2013). On the other hand, under H_1 of causality, the weak law of large numbers yields immediately

$$\begin{aligned}
D_T &= E\left[\rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})\right] - E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right] + o_p(1) \qquad (2) \\
&= E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right]\left(\frac{E\left[\rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})\right]}{E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right]} - 1\right) + o_p(1) \\
&= E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right]C_\tau(Y \to X) + o_p(1) \\
&:= \kappa(\tau) \times C_\tau(Y \to X) + o_p(1),
\end{aligned}$$

where the third step follows by a Taylor expansion of $\ln y$ around 1.

Therefore, since under H_1, $C_\tau(Y \to X) > 0$, or equivalently, $\Pr\left[q_\tau(\underline{Z}_{t-1}) = \bar{q}_\tau(\underline{X}_{t-1})\right] < 1$, we have

$$
\begin{aligned}
&Th_2^{d/2} C_\tau(\widehat{Y \to X}) \\
=&Th_2^{d/2} \frac{T^{-1}\sum_{t=1}^T \rho_\tau(X_t - \hat{\bar{\bar{q}}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - T^{-1}\sum_{t=1}^T \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})}{T^{-1}\sum_{t=1}^T \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})} \times [1 + o_p(1)] \\
=&\kappa(\tau)^{-1}\left[Th_2^{d/2} D_T + Th_2^{d/2}(A_T + B_T + C_T)\right] \times [1 + o_p(1)] \\
=&Th_2^{d/2} C_\tau(Y \to X) \to \infty.
\end{aligned}
$$

Alternatively, under H_1 of causality, one can simply apply the consistency result in Proposition 2 to show that $C_\tau(\widehat{Y \to X})$ converges in probability to $C_\tau(Y \to X) > 0$, and consequently $Th_2^{d/2} C_\tau(\widehat{Y \to X})$ will diverge to infinity under our assumptions.

On the other hand, following arguments similar to those we have used in the proof of the consistency of estimator $\widehat{\sigma}_{0\tau}^2$ to the asymptotic variance $\sigma_{0\tau}^2$ in Theorem 2 under the null hypothesis, we can show that $\widehat{\sigma}_{0\tau}^2 := \widehat{\bar{\sigma}}_{0\tau}^2 / \widehat{C}_\tau^2 = O_p(1)$ under the alternative hypothesis of no causality. Proposition 3 follows then from $Th_2^{d/2} C_\tau(\widehat{Y \to X}) \to \infty$ and $\widehat{\sigma}_{0\tau} = O_p(1)$ as $T \to \infty$. Hence, the test $\widehat{\Gamma}_\tau = Th_2^{d/2} C_\tau(\widehat{Y \to X}) / \widehat{\sigma}_{0\tau}$ is diverging to infinity at the rate $Th_2^{d/2}$ and is consistent. $\square$

Proof of Proposition 4: First, following similar arguments as in Theorem 2 and Proposition 3, with the only exception that the term D_T defined in (2) now takes a different form. Specifically, we can show that under the local alternatives given in Equation (18) of the main text,

$$
\begin{aligned}
&Th_2^{d/2}\left(\frac{1}{T}\sum_{t=1}^T \rho_\tau(X_t - \hat{\bar{\bar{q}}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - \frac{1}{T}\sum_{t=1}^T \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1}) - D_T\right) \\
:=&Th_2^{d/2}(A_T + B_T + C_T) \xrightarrow{d} \mathcal{N}(0, \tilde{\sigma}_{0\tau}^2),
\end{aligned}
$$

with A_T, B_T, C_T, and $\tilde{\sigma}_{0\tau}^2$ given in the proof of Theorem 2.

Second, under the local alternative hypotheses $H_1(\delta_T)$, using the second order Taylor expansion, one can calculate that

$$
\begin{aligned}
D_T &= E\left[\rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})\right] - E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right][1 + o_p(1)] \\
&= E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}) + \delta_T \Delta_T(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right] - E\left[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})\right][1 + o_p(1)] \\
&= \delta_T E\left[\Delta_T(\underline{Z}_{t-1})w(\underline{Z}_{t-1})\varphi(\varepsilon_t)\right] - \frac{\delta_T^2}{2} E\left[\Delta_T^2(\underline{Z}_{t-1})w(\underline{Z}_{t-1})g(\underline{Z}_{t-1})\right][1 + o_p(1)] \\
&= \delta_T^2 E\left[\Delta_T^2(\underline{Z}_{t-1})w(\underline{Z}_{t-1})f_{\varepsilon|\underline{Z}}(0|\underline{Z}_{t-1})\right][1 + o_p(1)],
\end{aligned}
$$

where $g(\underline{z}) = \partial E[\varphi(X_t - \theta)|\underline{Z}_{t-1} = \underline{z}]/\partial\theta = -f_{\varepsilon|\underline{Z}}(0|\underline{z})$, and the fourth step follows by law of iterated expectations and $E[\varphi(\varepsilon_t)|\underline{Z}_{t-1}] = 0$ in Assumption A.3. Consequently, with $\delta_T = \left(Th_2^{d/2}\right)^{-1/2}$, we have

$$\begin{aligned} Th_2^{d/2} C_\tau(\widehat{Y \to X}) &= \kappa(\tau)^{-1} \left[Th_2^{d/2} D_T + Th_2^{d/2}(A_T + B_T + C_T)\right] \times [1 + o_p(1)] \\ &\xrightarrow{d} \mathcal{N}\left(\gamma, \sigma_{0\tau}^2\right) \end{aligned}$$

under the local alternatives with

$$\gamma = \kappa(\tau)^{-1} \lim_{T\to\infty} E\left[\Delta_T^2(\underline{Z}_{t-1}) w(\underline{Z}_{t-1}) f_{\varepsilon|\underline{Z}}(0|\underline{Z}_{t-1})\right].$$

This concludes the proof of Proposition 4. □

Proof of Theorem 3: The asymptotic validity of our bootstrap procedure can be proved using similar arguments to those used in the proof of Theorem 2, with the term A_{1T} replaced by its bootstrapped version A_{1T}^* using the bootstrapped sample $\{(X_t^*, Y_t^*)\}_{t=1}^T$. Conditionally on $\{(X_t, Y_t)\}_{t=1}^T$ and using Theorem 1 of Hall (1984), we obtain the bootstrap validity result in Theorem 3. □

Proof of Theorem 4: The proof is similar to the proof of Theorem 1 and a sketched proof is provided.

Denote $\widehat{\overline{d}}_1(\underline{W}_{t-1}) = (\widehat{\overline{\phi}}_\tau - \overline{\phi}_\tau)'\underline{W}_{t-1}$ and $\widehat{\overline{d}}_2(\underline{X}_{t-1}) = \widehat{\overline{q}}_\tau(\underline{X}_{t-1}) - \bar{q}_\tau(\underline{X}_{t-1})$. Note the following expansion holds,

$$\begin{aligned} &\frac{1}{T}\sum_{t=1}^T \rho_\tau\left(X_t - \widehat{\overline{\phi}}'_\tau \underline{W}_{t-1} - \widehat{\overline{q}}_\tau(\underline{X}_{t-1})\right) w(\underline{Z}_{t-1}) \\ =&\frac{1}{T}\sum_{t=1}^T \rho_\tau\left(X_t - \overline{\phi}'_\tau \underline{W}_{t-1} - \bar{q}_\tau(\underline{X}_{t-1})\right) w(\underline{Z}_{t-1}) \\ &- \frac{1}{T}\sum_{t=1}^T \left(\widehat{\overline{q}}_\tau(\underline{X}_{t-1}) - \bar{q}_\tau(\underline{X}_{t-1})\right) w(\underline{Z}_{t-1})\varphi(\bar{\varepsilon}_t) - \frac{1}{T}\sum_{t=1}^T \left(\widehat{\overline{\phi}}_\tau - \overline{\phi}_\tau\right)' \underline{W}_{t-1} w(\underline{Z}_{t-1})\varphi(\bar{\varepsilon}_t) \\ &- \frac{2}{T}\sum_{t=1}^T \left(X_t - \widehat{\overline{\phi}}' \underline{W}_{t-1} - \widehat{\overline{q}}_\tau(\underline{X}_{t-1})\right)\Big\{1(\widehat{\overline{d}}_1(\underline{W}_{t-1}) + \widehat{\overline{d}}_2(\underline{X}_{t-1}) > \bar{\varepsilon}_t > 0) \\ &-1(\widehat{\overline{d}}_1(\underline{W}_{t-1}) + \widehat{\overline{d}}_2(\underline{X}_{t-1}) < \bar{\varepsilon}_t < 0)\Big\} w(\underline{Z}_{t-1}) \\ :=&\frac{1}{T}\sum_{t=1}^T \rho_\tau\left(X_t - \overline{\phi}'_\tau \underline{W}_{t-1} - \bar{q}_\tau(\underline{X}_{t-1})\right) w(\underline{Z}_{t-1}) + E_T + F_T + G_T \\ =&\frac{1}{T}\sum_{t=1}^T \rho_\tau\left(X_t - \overline{\phi}'_\tau \underline{W}_{t-1} - \bar{q}_\tau(\underline{X}_{t-1})\right) w(\underline{Z}_{t-1}) + o_p(T^{-1/2}), \end{aligned}$$

where $E_T = F_T = G_T = o_p(T^{-1/2})$ can be proved using the steps in the proof of lemma 3 in Noh et al. (2013) and by noting that $\max_{t\in I(w)} |\widehat{\overline{d}}_1(\underline{W}_{t-1})| = O_p(T^{-1/2})$ for bounded support. Moreover, the asymptotic representation for

$$\frac{1}{T}\sum_{t=1}^T \rho_\tau\left(X_t - \widehat{\phi}'_\tau \underline{W}_{t-1} - \hat{q}_\tau(\underline{Z}_{t-1})\right) w(\underline{Z}_{t-1}) = \frac{1}{T}\sum_{t=1}^T \rho_\tau\left(X_t - \phi'_\tau \underline{W}_{t-1} - q_\tau(\underline{Z}_{t-1})\right) w(\underline{Z}_{t-1}) + o_p(T^{-1/2})$$

can be obtained using the same arguments as Lemmas 3 and 4 of Appendix C. The proof then follows by using the equality that $\hat{a}\hat{b}^{-1} = ab^{-1} + \hat{b}^{-1}\left[(\hat{a}-a) - \left(\hat{b}-b\right)ab^{-1}\right]$. $\square$

Proof of Theorem 5: Consider the following decomposition of $\rho_\tau(\cdot)$:

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\rho_\tau\left(X_t - \widehat{\overline{\phi}}'_\tau \underline{W}_{t-1} - \widehat{\overline{q}}_\tau(\underline{X}_{t-1})\right)w(\underline{Z}_{t-1}) - \frac{1}{T}\sum_{t=1}^{T}\rho_\tau\left(X_t - \widehat{\phi}'_\tau \underline{W}_{t-1} - \hat{q}_\tau(\underline{Z}_{t-1})\right)w(\underline{Z}_{t-1})\\
=&\frac{1}{T}\sum_{t=1}^{T}\left[\left(\hat{q}_\tau(\underline{Z}_{t-1}) - q_\tau(\underline{Z}_{t-1})\right) - \left(\widehat{\overline{q}}_\tau(\underline{X}_{t-1}) - \bar{q}_\tau(\underline{X}_{t-1})\right)\right]w(\underline{Z}_{t-1})\varphi(\varepsilon_t)\\
&+\frac{1}{T}\sum_{t=1}^{T}\left(\left(\widehat{\phi}_\tau - \phi_\tau\right) - \left(\widehat{\overline{\phi}}_\tau - \overline{\phi}_\tau\right)\right)'\underline{W}_{t-1}w(\underline{Z}_{t-1})\varphi(\varepsilon_t) + \text{higher order terms}\\
:=&H_T + I_T + \text{higher order terms}.
\end{aligned}$$

Following the same arguments as in the proof of Theorem 2, it can be shown that $Th_2^{d/2}H_T \xrightarrow{d} \mathcal{N}(0, \tilde{\sigma}^2_{0\tau})$ and $Th_2^{d/2}I_T = \left[\sqrt{T}\left(\widehat{\phi}_\tau - \phi_\tau\right) - \sqrt{T}\left(\widehat{\overline{\phi}}_\tau - \overline{\phi}_\tau\right)\right]h_2^{d/2}O_p(1) = o_p(1)$ by root-T consistency properties of linear coefficients estimators $\widehat{\phi}_\tau$ and $\widehat{\overline{\phi}}_\tau$. Therefore,

$$\begin{aligned}
&Th_2^{d/2}C_\tau^{PL}\left(\widehat{Y \to X|W}\right)\\
=&(\kappa^{PL}(\tau))^{-1}Th_2^{d/2}H_T \times [1 + o_p(1)]\\
\xrightarrow{d}&\mathcal{N}(0, \sigma^{PL2}_{0\tau}),
\end{aligned}$$

which proves that $Th_2^{d/2}C_\tau^{PL}\left(\widehat{Y \to X|W}\right)$ converges to a normal distribution under the null of no causality in the presence of control variables W. $\square$

Appendix C: Proofs of auxiliary results

In this section, we provide four auxiliary lemmas which are useful to prove our main results in Appendix B. In the first lemma, the uniform Bahadur representation for the estimator of the restricted conditional quantile function $\bar{q}_\tau(\underline{x})$ based on a p-th order local polynomial approximation using bandwidth h_1 is derived. Please notice that the proofs of the following Lemmas 1-4 can be obtained using similar arguments as in lemmas 2 and 3 in Noh et al. (2013) [or using results in Kong et al. (2010), see their corollary 1, lemmas 8 and 10, respectively], and they are therefore omitted.

Lemma 1: *Let $\overline{e}_1$ be an $N_1 \times 1$ vector with its first element given by 1 and all others 0. Suppose* ***A.1-A.9*** *in Appendix A hold and $h_1 = O\left(T^{-\kappa_1}\right)$ with $\kappa_1 > 1/\left(2p+2+d_1\right)$. Then, with probability one, we have*

$$\widehat{\overline{q}}_\tau(\underline{x}) - \bar{q}_\tau(\underline{x}) = -\overline{e}'_1\frac{H_T^{-1}}{Th_1^{d_1}}S_{T,p}^{-1}(\underline{x})\sum_{t=1}^{T}K_{h_1}(\underline{X}_{t-1} - \underline{x})\varphi(\overline{\varepsilon}_t)\mu(\underline{X}_{t-1} - \underline{x}) + \overline{R}_T,$$

where $\overline{\varepsilon}_t = X_t - \bar{q}_\tau\left(\underline{X}_{t-1}\right)$ is the restricted error and $\overline{R}_T = o_p\left(\left(Th_1^{d_1}\right)^{-1/2}\right)$ uniformly in $\underline{x} \in \mathcal{D}_X$ and $\mathcal{D}_X$ is the compact support of the weighting function $w(\cdot)$ with respect to the part of X.

Analogously, the q-th order local polynomial estimator of the unrestricted conditional quantile function $q_\tau(\underline{z})$ using bandwidth h_2, say $\hat{q}_\tau(\underline{z})$, can be defined accordingly as in Section 4 and its uniform Bahadur representation can be obtained similarly and is stated in the next lemma. Note that Lemma 1 is only a special case of Lemma 2.

Lemma 2: *Denote $d = d_1 + d_2$. Let $\underline{e}_1$ be an $N_2 \times 1$ vector with its first element given by 1 and all others 0. Suppose Assumptions* ***A.1-A.9*** *in Appendix A hold and $h_2 = O(T^{-\kappa_2})$ with $\kappa_2 > 1/(2q+2+d)$. Then, with probability one, we have*

$$\hat{q}_\tau(\underline{z}) - q_\tau(\underline{z}) = -\underline{e}_1' \frac{H_T^{-1}}{Th_2^d} S_{T,q}^{-1}(\underline{z}) \sum_{t=1}^{T} K_{h_2}(\underline{Z}_{t-1} - \underline{z}) \varphi(\varepsilon_t) \mu(\underline{Z}_{t-1} - \underline{z}) + R_T,$$

where $\varepsilon_t = X_t - q_\tau(\underline{Z}_{t-1})$ is the unrestricted error and $R_T = o_p\left(\left(Th_2^d\right)^{-1/2}\right)$ uniformly in $\underline{z} \in \mathcal{D}$ and $\mathcal{D}$ is the compact support of the weighting function $w(\cdot)$.

On the other hand, to derive the Bahadur representation of $\widehat{C_\tau(Y \to X)}$, we need to investigate the asymptotic behaviour of $T^{-1}\sum_{t=1}^{T} \rho_\tau(X_t - \hat{\bar{q}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})$ [resp. $T^{-1}\sum_{t=1}^{T} \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})$], which is stated in the next two lemmas. Again, the proof of Lemma 3 is similar to the one of Lemma 4.

Lemma 3: *Suppose Assumptions* ***A.1-A.9*** *in Appendix A hold, $p > d_1/2 - 1$ and $h_1 = O(T^{-\kappa_1})$ with $1/(2p+2+d_1) < \kappa_1 < 1/(2d_1)$. Then,*

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} \rho_\tau(X_t - \hat{\bar{q}}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - E[\rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})] \\
=&\frac{1}{T}\sum_{t=1}^{T} \rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1}) - E[\rho_\tau(X_t - \bar{q}_\tau(\underline{X}_{t-1}))w(\underline{Z}_{t-1})] + o_p(T^{-1/2}).
\end{aligned}$$

Lemma 4: *Let $d = d_1 + d_2$. Suppose Assumptions* ***A.1-A.9*** *in Appendix A hold, $q > d/2 - 1$ and $h_2 = O(T^{-\kappa_2})$ with $1/(2q+2+d) < \kappa_2 < 1/(2d)$. Then, we have*

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} \rho_\tau(X_t - \hat{q}_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1}) - E[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})] \\
=&\frac{1}{T}\sum_{t=1}^{T} \rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1}) - E[\rho_\tau(X_t - q_\tau(\underline{Z}_{t-1}))w(\underline{Z}_{t-1})] + o_p(T^{-1/2}).
\end{aligned}$$

Appendix D: Additional simulation results

Table 1: Bootstrap bias-corrected estimation of quantile Granger causality measures at $\tau = 0.25$

	Measure	DGP S1	DGP S2	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5	DGP P6
				$T = 50$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.1122 (0.1152)	0.1094 (0.1112)	0.7501 (0.3056)	0.7635 (0.3169)	1.0212 (0.3718)	0.7162 (0.3283)	0.8587 (0.3792)	0.8246 (0.3589)
				$T = 100$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.0609 (0.0565)	0.0544 (0.0534)	0.6049 (0.1826)	0.6295 (0.1980)	0.8371 (0.2283)	0.6157 (0.2096)	0.6260 (0.2026)	0.6279 (0.2219)
				$T = 200$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.0380 (0.0316)	0.0314 (0.0319)	0.5071 (0.1138)	0.5370 (0.1250)	0.7156 (0.1458)	0.5649 (0.1476)	0.4696 (0.1206)	0.4889 (0.1305)
				$T = 400$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.0230 (0.0180)	0.0180 (0.0184)	0.4050 (0.0670)	0.4303 (0.0729)	0.5851 (0.0861)	0.4762 (0.0984)	0.3129 (0.0673)	0.3282 (0.0755)

Note: This table shows the average values of bootstrap bias-corrected ($\widehat{C^*_{\tau,BC}(Y \to X)}$) estimates of causality measures from Y to X ($C_\tau(Y \to X)$). "True" indicates the true value of causality measure, "Bias-Corrected" indicates the average value of the estimate of causality measure after bootstrap bias correction, and "—" means that the true value of causality measure is unknown. Equation (29) in the main text is used to calculate the bootstrap bias-correction estimates of causality measures. The number of simulations used to compute the averaged values of the estimates of causality measures and the number of bootstrap replications used to calculate the bias-corrected estimates are equal to 500 and 199, respectively. "No" indicates non-causality in the true DGP and "Yes" means that there is causality in the true GDP at the specified quantile. The DGPs in the first row of the table are described in detail in Table 1 of the main text. In parenthesis is the standard deviation of the estimated value.

Table 2: Bootstrap bias-corrected estimation of quantile Granger causality measures at $\tau = 0.50$

	Measure	DGP S1	DGP S2	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5	DGP P6
				$T = 50$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	No
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.1076 (0.1050)	0.0882 (0.0963)	0.6746 (0.3028)	0.6725 (0.3058)	0.9096 (0.3623)	0.6369 (0.3324)	0.7973 (0.3655)	0.1150 (0.1120)
				$T = 100$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	No
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.0584 (0.0578)	0.0510 (0.0536)	0.5335 (0.1632)	0.5453 (0.1658)	0.8045 (0.2457)	0.5695 (0.2080)	0.5597 (0.2038)	0.0654 (0.0648)
				$T = 200$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	No
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.0332 (0.0307)	0.0294 (0.0277)	0.4361 (0.0998)	0.4661 (0.1090)	0.7082 (0.1482)	0.5130 (0.1403)	0.4201 (0.1172)	0.0364 (0.0325)
				$T = 400$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	No
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$\widehat{C^*_{\tau,BC}(Y \to X)}$	0.0202 (0.0160)	0.0156 (0.0150)	0.3929 (0.0618)	0.4075 (0.0675)	0.6536 (0.1083)	0.4769 (0.1021)	0.3176 (0.0736)	0.0241 (0.0228)

Note: This table shows the average values of bootstrap bias-corrected ($\widehat{C^*_{\tau,BC}(Y \to X)}$) estimates of causality measures from Y to X ($C_\tau(Y \to X)$). "True" indicates the true value of causality measure, "Bias-Corrected" indicates the average value of the estimate of causality measure after bootstrap bias correction, and "—" means that the true value of causality measure is unknown. Equation (29) in the main text is used to calculate the bootstrap bias-correction estimates of causality measures. The number of simulations used to compute the averaged values of the estimates of causality measures and the number of bootstrap replications used to calculate the bias-corrected estimates are equal to 500 and 199, respectively. "No" indicates non-causality in the true DGP and "Yes" means that there is causality in the true GDP at the specified quantile. The DGPs in the first row of the table are described in detail in Table 1 of the main text. In parenthesis is the standard deviation of the estimated value.

Table 3: Bootstrap bias-corrected estimation of quantile Granger causality measures at $\tau = 0.75$

	Measure	DGP S1	DGP S2	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5	DGP P6
				$T = 50$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$C^*_{\tau,BC}\widehat{(Y \to X)}$	0.1062 (0.1090)	0.0891 (0.0995)	0.6985 (0.2831)	0.7092 (0.3075)	0.9373 (0.3925)	0.6840 (0.3078)	0.7700 (0.3706)	0.7603 (0.3410)
				$T = 100$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$C^*_{\tau,BC}\widehat{(Y \to X)}$	0.0633 (0.0601)	0.0624 (0.0601)	0.5595 (0.1828)	0.5699 (0.1751)	0.8810 (0.2648)	0.5672 (0.1953)	0.6344 (0.2160)	0.5550 (0.1961)
				$T = 200$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$C^*_{\tau,BC}\widehat{(Y \to X)}$	0.0396 (0.0311)	0.0339 (0.0312)	0.4582 (0.1068)	0.4899 (0.1166)	0.8340 (0.1857)	0.5217 (0.1415)	0.5183 (0.1322)	0.4274 (0.1192)
				$T = 400$					
$Y \to X$		No	No	Yes	Yes	Yes	Yes	Yes	Yes
True	$C_\tau(Y \to X)$	0.0000	0.0000	—	—	—	—	—	—
Bias-Corrected	$C^*_{\tau,BC}\widehat{(Y \to X)}$	0.0239 (0.0194)	0.0164 (0.0165)	0.4045 (0.0674)	0.4253 (0.0709)	0.7746 (0.1179)	0.4768 (0.1010)	0.4275 (0.0851)	0.3286 (0.0718)

Note: This table shows the average values of bootstrap bias-corrected ($C^*_{\tau,BC}\widehat{(Y \to X)}$) estimates of causality measures from Y to X ($C_\tau(Y \to X)$). "True" indicates the true value of causality measure, "Bias-Corrected" indicates the average value of the estimate of causality measure after bootstrap bias correction, and "—" means that the true value of causality measure is unknown. Equation (29) in the main text is used to calculate the bootstrap bias-correction estimates of causality measures. The number of simulations used to compute the averaged values of the estimates of causality measures and the number of bootstrap replications used to calculate the bias-corrected estimates are equal to 500 and 199, respectively. "No" indicates non-causality in the true DGP and "Yes" means that there is causality in the true GDP at the specified quantile. The DGPs in the first row of the table are described in detail in Table 1 of the main text. In parenthesis is the standard deviation of the estimated value.

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