

Supplementary material

for review of the manuscript

Properties and Approximate p -value Calculation of the Cramer test

1 Statistical properties of the test statistic $T_{n,m}$

Using the notation and definition as in the main manuscript, we present the proof for the expected value $E[T_{n,m}]$ and variance $\text{Var}[T_{n,m}]$ of the test statistic $T_{n,m}$ in the case of $n \neq m$.

Proposition 1 The expectation of the test statistic $T_{n,m}$ where $n > m$ follows the following formula;

$$E[T_{n,m}] = \int_{-\infty}^{\infty} (H(t) - H(t)^2) dt,$$

where $H(t)$ is the distribution of the two data sets under the null hypothesis.

Proof Consider the test statistic

$$T_{n,m} = \frac{nm}{n+m} \int_{-\infty}^{\infty} (\hat{F}(t) - \hat{G}(t))^2 dt \quad (1)$$

where

$$\begin{aligned} \hat{F}(t) &= \frac{1}{n} \sum_{i=1}^n 1_{\{x_i \leq t\}}; \\ \hat{G}(t) &= \frac{1}{m} \sum_{i=1}^m 1_{\{y_i \leq t\}}, \end{aligned}$$

the empirical cumulative distribution functions for distributions f and g respectively.

The formulas for the empirical cumulative distribution function can be substituted into equation (1) to give

$$T_{n,m} = \frac{nm}{n+m} \int_{-\infty}^{\infty} \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n 1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} - \frac{2}{nm} \sum_{i=1}^n \sum_{j=1}^m 1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} + \frac{1}{m^2} \sum_{i=1}^m \sum_{j=1}^m 1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} dt.$$

Now, the expectation of the test statistic can be calculated as follows;

$$\begin{aligned} E[T_{n,m}] &= \frac{nm}{n+m} \int_{-\infty}^{\infty} \left(\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n E[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}}] - \frac{2}{nm} \sum_{i=1}^n \sum_{j=1}^m E[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}}] \right. \\ &\quad \left. + \frac{1}{m^2} \sum_{i=1}^m \sum_{j=1}^m E[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}}] \right) dt \\ &= \frac{nm}{n+m} \int_{-\infty}^{\infty} \left(\frac{n-1}{n} E[1_{\{x_i \leq t\}}] E[1_{\{x_j \leq t\}}] + \frac{1}{n} E[1_{\{x_i \leq t\}}] \right. \\ &\quad \left. - 2 E[1_{\{x_i \leq t\}}] E[1_{\{y_j \leq t\}}] + \frac{m-1}{m} E[1_{\{y_i \leq t\}}] E[1_{\{y_j \leq t\}}] \right. \\ &\quad \left. + \frac{1}{m} E[1_{\{y_i \leq t\}}] \right) dt. \end{aligned}$$

Now, under the null hypothesis, the data points $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_m$ are assumed to follow the same distribution, namely $H(t)$. Hence

$$\mathbb{E}[1_{\{x_i \leq t\}}] = \mathbb{E}[1_{\{y_i \leq t\}}] = H(t).$$

Thus,

$$\mathbb{E}[T_{n,m}] = \int_{-\infty}^{\infty} (H(t) - H(t)^2) dt.$$

□

Proposition 2 The variance of the test statistic $T_{n,m}$ where $n > m$ follows the following formula;

$$\begin{aligned} \text{Var}[T_{n,m}] = & \frac{2(n^3 + m^3)}{nm(n+m)^2} \int_{-\infty}^{\infty} \int_{-\infty}^t H(s) \left(1 + 2 \left(\frac{nm(n+m)^2}{n^3 + m^3} - 2 \right) H(s) - 3H(t) \right. \\ & \left. - 2 \left(\frac{2nm(n+m)^2}{n^3 + m^3} - 5 \right) H(s)H(t) + 2H(t)^2 + 2 \left(\frac{nm(n+m)^2}{n^3 + m^3} - 3 \right) H(s)H(t)^2 \right) ds dt \end{aligned}$$

where $H(t)$ is the distribution of the two data sets under the null hypothesis.

Proof Consider the test statistic

$$T_{n,m} = \frac{nm}{n+m} \int_{-\infty}^{\infty} (\hat{F}(t) - \hat{G}(t))^2 dt \quad (2)$$

where

$$\hat{F}(t) = \frac{1}{n} \sum_{i=1}^n 1_{\{x_i \leq t\}}$$

and

$$\hat{G}(t) = \frac{1}{m} \sum_{i=1}^m 1_{\{y_i \leq t\}}$$

are the empirical cumulative distribution functions for the distributions f and g , respectively.

From the standard theory,

$$\text{Var}[T_{n,m}] = \mathbb{E}[T_{n,m}^2] - \{\mathbb{E}[T_{n,m}]\}^2$$

and we have the expression for $\mathbb{E}[T_{n,m}]$ from Proposition 1. We now focus to obtain the expression for $\mathbb{E}[T_{n,m}^2]$.

Now,

$$\begin{aligned}
\mathbb{E}[T_{n,m}^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\hat{F}(t) - \hat{G}(t))^2 (\hat{F}(s) - \hat{G}(s))^2 ds dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E} \left[\frac{1}{n^4} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n 1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right. \\
&\quad - \frac{2}{n^3 m} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m 1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \\
&\quad + \frac{1}{n^2 m^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m 1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \\
&\quad - \frac{2}{n^3 m} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m 1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \\
&\quad + \frac{4}{n^2 m^2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m 1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \\
&\quad - \frac{2}{nm^3} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m 1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \\
&\quad + \frac{1}{n^2 m^2} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n 1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \\
&\quad - \frac{2}{nm^3} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m 1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \\
&\quad \left. + \frac{1}{m^4} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m 1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt. \tag{3}
\end{aligned}$$

Each term in equation (3) can be calculated separately. For the first term in equation (3), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{n^4} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\
&\quad \frac{1}{n^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ n \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_i \leq s\}} \right] + 2n(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_i \leq s\}} 1_{\{x_j \leq s\}} \right] \right. \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_i \leq s\}}^2 \right] + n(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_j \leq s\}}^2 \right] \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{x_j \leq t\}} 1_{\{x_i \leq s\}} \right] \\
&\quad + n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_j \leq s\}} 1_{\{x_k \leq s\}} \right] \\
&\quad + n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}}^2 \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{x_j \leq s\}} 1_{\{x_k \leq t\}} \right] \\
&\quad \left. + n(n-1)(n-2)(n-3) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] \right\} ds dt. \tag{4}
\end{aligned}$$

Now, equation (4) is equivalent to

$$\begin{aligned}
& \frac{1}{n^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\
& \frac{1}{n^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ n \text{pr}\{X_i \leq \min(s, t)\} + 2n(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_i \leq s\} \right. \\
& + 2n(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq \min(s, t)\} + n(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \\
& + 2n(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_j \leq \min(s, t)\} \\
& + n(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{X_k \leq s\} \\
& + n(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq t\} \text{pr}\{X_k \leq s\} \\
& + 4n(n-1)(n-2) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_j \leq t\} \text{pr}\{X_k \leq s\} \\
& \left. + n(n-1)(n-2)(n-3) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq t\} \text{pr}\{X_k \leq s\} \text{pr}\{X_l \leq s\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& \frac{1}{n^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\
& \frac{1}{n^4} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ nH(s) + 4n(n-1)H(s)^2 + 3n(n-1)H(s)H(t) \right. \right. \\
& \quad + 5n(n-1)(n-2)H(s)^2H(t) + n(n-1)(n-2)H(s)H(t)^2 \\
& \quad \left. \left. + n(n-1)(n-2)(n-3)H(s)^2H(t)^2 \right\} ds \right. \\
& \quad + \int_t^{\infty} \left\{ nH(t) + 4n(n-1)H(t)^2 + 3n(n-1)H(s)H(t) \right. \\
& \quad + n(n-1)(n-2)H(s)^2H(t) + 5n(n-1)(n-2)H(s)H(t)^2 \\
& \quad \left. \left. + n(n-1)(n-2)(n-3)H(s)^2H(t)^2 \right\} ds \right\} dt.
\end{aligned}$$

Next, the second term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{2}{n^3 m} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{n^3 m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_i \leq s\}} 1_{\{y_i \leq s\}} \right] + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_i \leq s\}} 1_{\{y_j \leq s\}} \right] \right. \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{y_i \leq s\}} 1_{\{x_j \leq s\}} \right] + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_i \leq s\}} 1_{\{x_j \leq t\}} \right] \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_j \leq s\}} 1_{\{y_j \leq s\}} \right] + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{x_j \leq t\}} 1_{\{y_j \leq s\}} \right] \\
& + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{x_j \leq s\}} 1_{\{y_k \leq s\}} \right] \\
& + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} \right] \\
& + 3m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} \right] \\
& \left. + m(n-1)(n-2)(n-3) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] \right\} ds dt. \tag{5}
\end{aligned}$$

Now, equation (5) is equivalent to

$$\begin{aligned}
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_i \leq s\} \right. \\
& + m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq s\} \\
& + m(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq s\} \text{pr}\{X_j \leq s\} \\
& + 2m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_i \leq s\} \text{pr}\{X_j \leq t\} \\
& + m(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_j \leq s\} \\
& + 2m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_j \leq s\} \\
& + m(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq s\} \\
& + 2m(n-1)(n-2) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& + 3m(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} \text{pr}\{X_j \leq t\} \text{pr}\{X_k \leq s\} \\
& \left. + m(n-1)(n-2)(n-3) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq t\} \text{pr}\{X_k \leq s\} \text{pr}\{Y_l \leq s\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ mnH(s)^2 + 6m(n-1)H(s)^2H(t) + 3m(n-1)(n-2)H(s)^2H(t) \right. \right. \\
& \quad \left. \left. + 3m(n-1)(n-2)H(s)^2H(t)^2 + m(n-1)(n-2)(n-3)H(s)^2H(t)^2 \right\} ds \right. \\
& \quad \left. + \int_t^{\infty} \left\{ nmH(t)H(s) + 2m(n-1)H(s)^2H(t) + m(n-1)(n-2)H(s)^2H(t) \right. \right. \\
& \quad \left. \left. + 4m(n-1)H(s)H(t)^2 + 2m(n-1)(n-2)H(t)^2H(s) \right. \right. \\
& \quad \left. \left. + 3m(n-1)(n-2)H(s)^2H(t)^2 + m(n-1)(n-2)(n-3)H(s)^2H(t)^2 \right\} ds \right\} dt.
\end{aligned}$$

Next, the third term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{n^2m^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt \\
& \frac{1}{n^2m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{y_i \leq s\}}^2 \right] + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{y_i \leq s\}} 1_{\{y_j \leq s\}} \right] \right. \\
& + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_j \leq s\}}^2 \right] + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{y_j \leq s\}}^2 \right] \\
& + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{x_j \leq t\}} 1_{\{y_j \leq s\}} \right] \\
& + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}}^2 1_{\{y_j \leq s\}} 1_{\{y_k \leq s\}} \right] \\
& + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}}^2 \right] \\
& + 4m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} \right] \\
& \left. + m(n-1)(n-2)(n-3) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] \right\} ds dt. \tag{6}
\end{aligned}$$

Now, equation (6) is equivalent to

$$\begin{aligned}
& \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} + 2m(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} \text{pr}\{Y_j \leq s\} \right. \\
& + 2m(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_j \leq s\} + m(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq s\} \\
& + 2m(n-1) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_j \leq s\} \\
& + m(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq s\} \text{pr}\{Y_k \leq s\} \\
& + m(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& + 4m(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& \left. + m(n-1)(n-2)(n-3) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_k \leq s\} \text{pr}\{Y_l \leq s\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ nmH(s)H(t) + 2m(m-1)H(s)^2H(t) + 2m(n-1)H(s)H(t)^2 \right. \right. \\
& + m(m-1)(n-2)H(s)^2H(t) + m(n-1)(n-2)H(s)H(t)^2 \\
& + 2m(m-1)H(t)^2H(s)^2 + 4m(m-1)(n-2)H(t)^2H(s)^2 \\
& \left. \left. + m(m-1)(n-2)(n-3)H(t)^2H(s)^2 \right\} ds \right. \\
& \left. \int_t^{\infty} \left\{ nmH(s)H(t) + 2m(m-1)H(s)^2H(t) + 2m(n-1)H(s)H(t)^2 \right. \right. \\
& + m(m-1)(n-2)H(s)^2H(t) + m(n-1)(n-2)H(s)H(t)^2 \\
& + 2m(m-1)H(t)^2H(s)^2 + 4m(m-1)(n-2)H(t)^2H(s)^2 \\
& \left. \left. + m(m-1)(n-2)(n-3)H(t)^2H(s)^2 \right\} ds \right\} dt.
\end{aligned}$$

Next, the fourth term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{2}{n^3 m} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\
& -\frac{2}{n^3 m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} \right] + m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} \right] \right. \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{y_i \leq t\}} 1_{\{x_j \leq t\}} \right] + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} 1_{\{x_j \leq s\}} \right] \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{x_j \leq t\}} 1_{\{y_j \leq t\}} \right] + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_j \leq t\}} \right] \\
& + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{x_j \leq t\}} 1_{\{y_k \leq t\}} \right] \\
& + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_k \leq t\}} \right] \\
& + 3m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{y_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{x_k \leq t\}} \right] \\
& \left. + m(n-1)(n-2)(n-3) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_j \leq s\}} 1_{\{x_k \leq t\}} 1_{\{y_l \leq t\}} \right] \right\} ds dt. \tag{7}
\end{aligned}$$

Now, equation (7) is equivalent to

$$\begin{aligned}
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_i \leq t\} \right. \\
& + m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \\
& + m(n-1) \text{pr}\{X_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_j \leq t\} \\
& + 2m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \\
& + m(n-1) \text{pr}\{X_i \leq s\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_j \leq t\} \\
& + 2m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_j \leq t\} \\
& + m(n-1)(n-2) \text{pr}\{X_i \leq s\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_k \leq t\} \\
& + 2m(n-1)(n-2) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq t\} \\
& + 3m(n-1)(n-2) \text{pr}\{X_i \leq s\} \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{X_k \leq t\} \\
& \left. + m(n-1)(n-2)(n-3) \text{pr}\{X_i \leq s\} \text{pr}\{X_j \leq s\} \text{pr}\{X_k \leq t\} \text{pr}\{Y_l \leq t\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& -\frac{2}{n^3m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -2\mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\
& \int_{-\infty}^{\infty} \left\{ \int_t^{\infty} \left\{ mnH(s)H(t) + 2m(n-1)H(s)H(t)^2 + m(n-1)(n-2)H(s)H(t)^2 \right. \right. \\
& + 4m(n-1)H(s)^2H(t) + 2m(n-1)(n-2)H(s)^2H(t) \\
& + 3m(n-1)(n-2)H(s)^2H(t)^2 + m(n-1)(n-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} \\
& + \int_{-\infty}^t \left\{ mnH(t)^2 + 6m(n-1)H(s)H(t)^2 + 3m(n-1)(n-2)H(s)H(t)^2 \right. \\
& + 3m(n-1)(n-2)H(s)^2H(t)^2 + m(n-1)(n-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} dt.
\end{aligned}$$

Next, the fifth term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4}{n^2 m^2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{4}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_i \leq s\}} \right] \right. \\
& + m(m-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_j \leq s\}} \right] \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{x_j \leq s\}} \right] \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_j \leq s\}} \right] \\
& + m(m-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq t\}} \right] \\
& + m(m-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_j \leq s\}} \right] \\
& + m(n-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{y_j \leq s\}} \right] \\
& + m(m-1) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{x_j \leq s\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_k \leq s\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_k \leq s\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} \right] \\
& + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_k \leq t\}} 1_{\{y_k \leq s\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq s\}} 1_{\{y_k \leq t\}} 1_{\{x_k \leq s\}} \right] \\
& \left. + m(m-1)(n-2)(n-3) \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] \right\} ds dt. \tag{8}
\end{aligned}$$

Now, equation (8) is equivalent to

$$\begin{aligned}
& \frac{4}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{4}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_i \leq \min(s, t)\} \right. \\
& + m(m-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_j \leq s\} \\
& + m(n-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \\
& + m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_i \leq s\} \\
& + m(m-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq s\} \\
& + m(m-1) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_j \leq s\} \\
& + m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq \min(s, t)\} \\
& + m(m-1) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_j \leq s\} \\
& + m(m-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq s\} \\
& + m(m-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_k \leq s\} \text{pr}\{Y_k \leq s\} \\
& + m(m-1)(n-2) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& + m(n-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq \min(s, t)\} \\
& + m(m-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{Y_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_k \leq s\} \\
& + m(m-1)(n-2) \text{pr}\{X_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq t\} \text{pr}\{X_k \leq s\} \\
& \left. + m(m-1)(n-2)(n-3) \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_k \leq s\} \text{pr}\{Y_l \leq s\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& \frac{4}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \int_{-\infty}^{\infty} \left\{ \int_t^{\infty} \left\{ nmH(s)^2 + 2m(n-1)H(s)^2H(t) + 2m(m-1)H(s)^2H(t) \right. \right. \\
& + 2m(m-1)H(s)^2H(t)^2 + m(n-1)(n-2)H(s)^2H(t) + m(m-1)(n-2)H(s)^2H(t) \\
& + 4m(m-1)(n-2)H(s)^2H(t)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} \\
& + \int_{-\infty}^{\infty} \left\{ \int_t^{\infty} \left\{ nmH(t)^2 + 2m(n-1)H(s)H(t)^2 + 2m(m-1)H(s)H(t)^2 \right. \right. \\
& + 2m(m-1)H(s)^2H(t)^2 + m(n-1)(n-2)H(s)H(t)^2 + m(m-1)(n-2)H(s)H(t)^2 \\
& + 4m(m-1)(n-2)H(s)^2H(t)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} dt.
\end{aligned}$$

Next, the sixth term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{2}{nm^3} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{y_i \leq s\}}^2 1_{\{y_i \leq t\}} 1_{\{x_i \leq t\}} \right] + m(n-1) \mathbb{E} \left[1_{\{y_i \leq s\}}^2 1_{\{y_i \leq t\}} 1_{\{x_j \leq t\}} \right] \right. \\
& + m(m-1) \mathbb{E} \left[1_{\{y_i \leq s\}}^2 1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} \right] + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq s\}} 1_{\{y_i \leq t\}} 1_{\{x_i \leq t\}} 1_{\{y_j \leq s\}} \right] \\
& + m(m-1) \mathbb{E} \left[1_{\{y_i \leq s\}}^2 1_{\{y_j \leq t\}} 1_{\{x_j \leq t\}} \right] + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq s\}} 1_{\{y_i \leq t\}} 1_{\{y_j \leq s\}} 1_{\{x_j \leq t\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{y_i \leq s\}}^2 1_{\{y_j \leq t\}} 1_{\{x_k \leq t\}} \right] \\
& + 2m(m-1)(n-2) \mathbb{E} \left[1_{\{y_i \leq s\}} 1_{\{y_i \leq t\}} 1_{\{y_j \leq s\}} 1_{\{x_k \leq t\}} \right] \\
& + 3m(m-1)(m-2) \mathbb{E} \left[1_{\{y_i \leq s\}} 1_{\{x_i \leq t\}} 1_{\{y_j \leq s\}} 1_{\{y_k \leq t\}} \right] \\
& \left. + m(m-1)(m-2)(n-3) \mathbb{E} \left[1_{\{y_i \leq s\}} 1_{\{y_j \leq s\}} 1_{\{y_k \leq t\}} 1_{\{x_l \leq t\}} \right] \right\} ds dt. \tag{9}
\end{aligned}$$

Now, equation (9) is equivalent to

$$\begin{aligned}
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{X_i^1 \leq \min(s, t)\} \text{pr}\{Y_i \leq t\} \right. \\
& + m(n-1) \text{pr}\{X_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \\
& + m(m-1) \text{pr}\{Y_i \leq s\} \text{pr}\{X_j \leq t\} \text{pr}\{Y_j \leq t\} \\
& + 2m(m-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq s\} \\
& + m(m-1) \text{pr}\{Y_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_j \leq t\} \\
& + 2m(m-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_j \leq s\} \text{pr}\{X_j \leq t\} \\
& + m(m-1)(n-2) \text{pr}\{Y_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_k \leq t\} \\
& + 2m(m-1)(n-2) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_j \leq s\} \text{pr}\{X_k \leq t\} \\
& + 3m(m-1)(m-2) \text{pr}\{Y_i \leq s\} \text{pr}\{X_i \leq t\} \text{pr}\{Y_j \leq s\} \text{pr}\{Y_k \leq t\} \\
& \left. + m(m-1)(m-2)(n-3) \text{pr}\{Y_i \leq s\} \text{pr}\{Y_j \leq s\} \text{pr}\{Y_k \leq t\} \text{pr}\{X_l \leq t\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{x_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \left\{ \int_t^{\infty} \left\{ mnH(s)H(t) + 2m(m-1)H(s)H(t)^2 \right. \right. \\
& + m(m-1)(n-2)H(s)H(t)^2 + 2m(m-1)(n-2)H(s)^2H(t) \\
& + 3m(m-1)(m-2)H(s)^2H(t)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} \\
& + \int_{-\infty}^t \left\{ mnH(t)^2 + 6m(m-1)H(s)H(t)^2 + 3m(m-1)(n-2)H(s)H(t)^2 \right. \\
& \left. + 3m(m-1)(m-2)H(s)^2H(t)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2 \right\} ds \left. \right\} dt.
\end{aligned}$$

Next, the seventh term in equation (3) can be calculated as

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{n^2 m^2} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\ & \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{y_i \leq t\}}^2 \right] + 2m(m-1) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} \right] \right. \\ & + 2m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_j \leq s\}} 1_{\{y_j \leq t\}}^2 \right] + m(n-1) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{y_j \leq t\}}^2 \right] \\ & + 2m(m-1) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{y_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_j \leq s\}} \right] \end{aligned} \quad (10)$$

$$\begin{aligned} & + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq s\}}^2 1_{\{y_j \leq t\}} 1_{\{y_k \leq t\}} \right] \\ & + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_j \leq s\}} 1_{\{y_k \leq t\}}^2 \right] \\ & + 4m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{y_i \leq t\}} 1_{\{x_j \leq s\}} 1_{\{y_k \leq t\}} \right] \\ & \left. + m(m-1)(n-2)(n-3) \mathbb{E} \left[1_{\{x_i \leq s\}} 1_{\{x_j \leq s\}} 1_{\{y_k \leq t\}} 1_{\{y_l \leq t\}} \right] \right\} ds dt. \end{aligned} \quad (11)$$

Now, equation (11) is equivalent to

$$\begin{aligned} & \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\ & \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{X_i \leq s\} \text{pr}\{Y_i \leq t\} + 2m(m-1) \text{pr}\{X_i \leq s\} \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq t\} \right. \\ & + 2m(n-1) \text{pr}\{X_i \leq s\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_j \leq t\} + m(n-1) \text{pr}\{X_i \leq s\} \text{pr}\{Y_j \leq t\} \\ & + 2m(n-1) \text{pr}\{X_i \leq s\} \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_j \leq t\} \\ & + m(m-1)(n-2) \text{pr}\{X_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq t\} \\ & + m(n-1)(n-2) \text{pr}\{X_i \leq s\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq t\} \\ & + 4m(m-1)(n-2) \text{pr}\{X_i \leq s\} \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq t\} \\ & \left. + m(m-1)(n-2)(n-3) \text{pr}\{X_i \leq s\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_k \leq t\} \text{pr}\{Y_l \leq t\} \right\} ds dt, \end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned} & \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{x_l \leq s\}} \right] ds dt = \\ & \frac{1}{n^2 m^2} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ mnH(s)H(t) + 2m(m-1)H(s)H(t)^2 + 2m(n-1)H(s)^2H(t) \right. \right. \\ & + 2m(m-1)H(s)^2H(t)^2 + m(m-1)(n-2)H(s)H(t)^2 + m(n-1)(n-2)H(s)^2H(t) \\ & + 4m(m-1)(n-2)H(s)^2H(t)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \\ & + \int_{-\infty}^t \left\{ mnH(s)H(t) + 2m(m-1)H(s)H(t)^2 + 2m(n-1)H(s)^2H(t) \right. \\ & + 2m(m-1)H(s)^2H(t)^2 + m(m-1)(n-2)H(s)H(t)^2 + m(n-1)(n-2)H(s)^2H(t) \\ & + 4m(m-1)(n-2)H(s)^2H(t)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} dt. \end{aligned}$$

Next, the eighth term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{2}{nm^3} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_i \leq s\}} 1_{\{x_i \leq s\}} \right] + m(n-1) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_i \leq s\}} 1_{\{x_j \leq s\}} \right] \right. \\
& + m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{x_i \leq s\}} 1_{\{y_j \leq s\}} \right] + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{x_i \leq s\}} 1_{\{y_j \leq t\}} \right] \\
& + m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_j \leq s\}} 1_{\{x_j \leq s\}} \right] + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{x_j \leq s\}} \right] \\
& + m(m-1)(n-2) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_j \leq s\}} 1_{\{x_k \leq s\}} \right] \\
& + 2m(m-1)(n-2) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} \right] \\
& + 3m(m-1)(m-2) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{x_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} \right] \\
& \left. + m(m-1)(m-2)(n-3) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{x_l \leq s\}} \right] \right\} ds dt. \tag{12}
\end{aligned}$$

Now, equation (12) is equivalent to

$$\begin{aligned}
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{X_i \leq s\} \right. \\
& + m(n-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{X_j \leq s\} \\
& + m(m-1) \text{pr}\{Y_i \leq t\} \text{pr}\{X_j \leq s\} \text{pr}\{Y_j \leq s\} \\
& + 2m(m-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{X_i \leq s\} \text{pr}\{Y_j \leq t\} \\
& + m(m-1) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq s\} \text{pr}\{X_j \leq s\} \\
& + 2m(m-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_j \leq s\} \\
& + m(m-1)(n-2) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq s\} \text{pr}\{X_k \leq s\} \\
& + 2m(m-1)(n-2) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \text{pr}\{X_k \leq s\} \\
& + 3m(m-1)(m-2) \text{pr}\{Y_i \leq t\} \text{pr}\{X_i \leq s\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& \left. + m(m-1)(m-2)(n-3) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq s\} \text{pr}\{X_l \leq s\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{x_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& -\frac{2}{nm^3} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ mnH(s)^2 + 6m(m-1)H(s)^2H(t) + 3m(m-1)(n-2)H(s)^2H(t) \right. \right. \\
& \quad \left. \left. + 3m(m-1)(m-2)H(s)^2H(t)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2 \right\} ds \right. \\
& \quad \left. + \int_t^{\infty} \left\{ mnH(s)H(t) + 2m(m-1)H(s)^2H(t) \right. \right. \\
& \quad \left. \left. + 4m(m-1)H(s)H(t)^2 + m(m-1)(n-2)H(s)^2H(t) + 2m(m-1)(n-2)H(s)H(t)^2 \right. \right. \\
& \quad \left. \left. + 3m(m-1)(m-2)H(s)^2H(t)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2 \right\} ds \right\} dt.
\end{aligned}$$

Next, the ninth term in equation (3) can be calculated as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{m^4} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{1}{m^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_i \leq s\}}^2 \right] + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_i \leq s\}} 1_{\{y_j \leq s\}} \right] \right. \\
& + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_i \leq s\}}^2 \right] + m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_j \leq s\}}^2 \right] \\
& + 2m(m-1) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq t\}} 1_{\{y_i \leq s\}} \right] \\
& + m(m-1)(m-2) \mathbb{E} \left[1_{\{y_i \leq t\}}^2 1_{\{y_j \leq s\}} 1_{\{y_k \leq s\}} \right] \\
& + m(m-1)(m-2) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}}^2 \right] \\
& + 4m(m-1)(m-2) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_i \leq s\}} 1_{\{y_j \leq s\}} 1_{\{y_k \leq t\}} \right] \\
& \left. + m(m-1)(m-2)(m-3) \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] \right\} ds dt. \tag{13}
\end{aligned}$$

Now, equation (13) is equivalent to

$$\begin{aligned}
& \frac{1}{m^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{1}{m^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ m \text{pr}\{Y_i \leq \min(s, t)\} + 2m(m-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_i \leq s\} \right. \\
& + 2m(m-1) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq \min(s, t)\} + m(m-1) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq s\} \\
& + 2m(m-1) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_j \leq \min(s, t)\} \\
& + m(m-1)(m-2) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq s\} \text{pr}\{Y_k \leq s\} \\
& + m(m-1)(m-2) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& + 4m(m-1)(m-2) \text{pr}\{Y_i \leq \min(s, t)\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq s\} \\
& \left. + m(m-1)(m-2)(m-3) \text{pr}\{Y_i \leq t\} \text{pr}\{Y_j \leq t\} \text{pr}\{Y_k \leq s\} \text{pr}\{Y_l \leq s\} \right\} ds dt,
\end{aligned}$$

which, under the null hypothesis, is equivalent to

$$\begin{aligned}
& \frac{1}{m^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \mathbb{E} \left[1_{\{y_i \leq t\}} 1_{\{y_j \leq t\}} 1_{\{y_k \leq s\}} 1_{\{y_l \leq s\}} \right] ds dt = \\
& \frac{1}{m^4} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ mH(s) + 4m(m-1)H(s)^2 + 3m(m-1)H(s)H(t) \right. \right. \\
& + 5m(m-1)(m-2)H(s)^2H(t) + m(m-1)(m-2)H(s)H(t)^2 \\
& + m(m-1)(m-2)(m-3)H(s)^2H(t)^2 \left. \right\} ds \\
& + \int_t^{\infty} \left\{ mH(t) + 4m(m-1)H(t)^2 + 3m(m-1)H(s)H(t) \right. \\
& + m(m-1)(m-2)H(s)^2H(t) + 5m(m-1)(m-2)H(s)H(t)^2 \\
& + m(m-1)(m-2)(m-3)H(s)^2H(t)^2 \left. \right\} ds \left. \right\} dt.
\end{aligned}$$

Collecting all the calculated terms from equation (3), we get

$$\begin{aligned} E[T_{n,m}^2] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t \left\{ \frac{2(n^3 + m^3)}{nm(n+m)^2} H(s) + \left(4 - \frac{8(n^3 + m^3)}{nm(n+m)^3} \right) H(s)^2 \right. \right. \\ + \left(2 - \frac{6(n^3 + m^3)}{nm(n+m)^2} \right) H(s)H(t) - \left(10 - \frac{20(n^3 + m^3)}{nm(n+m)^2} \right) H(s)^2 H(t) \\ - \left(2 - \frac{4(n^3 + m^3)}{nm(n+m)^2} \right) H(s)H(t)^2 \\ \left. \left. + \left(6 - \frac{12(n^3 + m^3)}{nm(n+m)^2} \right) H(s)^2 H(t)^2 \right\} ds \right\} dt. \end{aligned} \quad (14)$$

Now, from Proposition 1,

$$E[T_{n,m}] = \int_{-\infty}^{\infty} (H(t) - H(t)^2) dt,$$

and thus

$$\begin{aligned} \{E[T_{n,m}]\}^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(t)(1 - H(t))H(s)(1 - H(s))ds dt \\ &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^t H(s)H(t)(1 - H(s))(1 - H(t))ds dt \\ &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^t (H(s)H(t) - H(s)H(t)^2 - H(s)^2 H(t) + H(s)^2 H(t)^2) ds dt. \end{aligned} \quad (15)$$

Collecting together equations (14) and (15) we get

$$\begin{aligned} \text{Var}[T_{n,m}] &= \int_{-\infty}^{\infty} \int_{-\infty}^t \frac{2(n^3 + m^3)}{nm(n+m)^2} H(s) \\ &+ \left(4 - \frac{8(n^3 + m^3)}{nm(n+m)^2} \right) H(s)^2 - \frac{6(n^3 + m^3)}{nm(n+m)^2} H(s)H(t) - \left(8 - \frac{20(n^3 + m^3)}{nm(n+m)^2} \right) H(s)^2 H(t) \\ &+ \frac{4(n^3 + m^3)}{nm(n+m)^2} H(s)H(t)^2 + \left(4 - \frac{12(n^3 + m^3)}{nm(n+m)^2} \right) H(s)^2 H(t)^2 ds dt, \end{aligned}$$

which can be factorized to give

$$\begin{aligned} \text{Var}[T_{n,m}] &= \frac{2(n^3 + m^3)}{nm(n+m)^2} \int_{-\infty}^{\infty} \int_{-\infty}^t H(s) \left(1 + 2 \left(\frac{nm(n+m)^2}{n^3 + m^3} - 2 \right) H(s) - 3H(t) \right. \\ &\quad \left. - 2 \left(\frac{2nm(n+m)^2}{n^3 + m^3} - 5 \right) H(s)H(t) + 2H(t)^2 + 2 \left(\frac{nm(n+m)^2}{n^3 + m^3} - 3 \right) H(s)H(t)^2 \right) ds dt. \end{aligned}$$

□

Proposition 3 The third moment of the test statistic $T_{n,m}$ where $n > m$ follows the following formula;

$$\begin{aligned}
E[T_{n,m}^3|F=G] = & \frac{6(n^5+m^5)}{(n+m)^3(nm)^2} \int_{-\infty}^{\infty} \int_{-\infty}^r \int_{-\infty}^s H(t) \left(1 \right. \\
& + 2 \left(\frac{nm(n+m)^2(7m^2-10nm+7n^2)}{n^5+m^5} - 8 \right) H(t) \\
& + 2 \left(\frac{nm(n+m)^2(5m^2-7nm+5n^2)}{n^5+m^5} - 6 \right) H(s) + \left(\frac{nm(n+m)^2(m^2-nm+n^2)}{n^5+m^5} - 3 \right) H(r) \\
& + 5 \left(\frac{nm(n+m)^2(2m^2n+2mn^2-19m^2+25mn-19n^2)}{n^5+m^5} + 18 \right) H(s)H(t) \\
& + \left(\frac{nm(n+m)^2(2m^2n+2mn^2-19m^2+25mn-19n^2)}{n^5+m^5} + 18 \right) H(s)^2 \\
& - \left(\frac{nm(n+m)^2(m^2-nm+n^2)}{n^5+m^5} - 2 \right) H(r)^2 \\
& + 2 \left(\frac{nm(n+m)^2(m^2n+mn^2-19m^2+26mn-19n^2)}{n^5+m^5} + 20 \right) H(r)H(t) \\
& + \left(\frac{nm(n+m)^2(m^2n+mn^2-27m^2+37mn-27n^2)}{n^5+m^5} + 30 \right) H(r)H(s) \\
& - 4 \left(\frac{nm(n+m)^2(3m^2n+3mn^2-26m^2+34mn-26n^2)}{n^5+m^5} + 24 \right) H(s)^2H(t) \\
& - 5 \left(\frac{nm(n+m)^2(5m^2n+5mn^2-45m^2+59mn-49n^2)}{n^5+m^5} + 42 \right) H(r)H(s)H(t) \\
& - 2 \left(\frac{nm(n+m)^2(m^2n+mn^2-12m^2+16mn-12n^2)}{n^5+m^5} + 12 \right) H(r)^2H(t) \\
& - \left(\frac{nm(n+m)^2(5m^2n+5mn^2-45m^2+59mn-49n^2)}{n^5+m^5} + 42 \right) H(r)H(s)^2 \\
& - \left(\frac{nm(n+m)^2(m^2n+mn^2-17m^2+23mn-17n^2)}{n^5+m^5} + 18 \right) H(r)^2H(s) \\
& + 9 \left(\frac{nm(n+m)^2(3m^2n+3mn^2-26m^2+34mn-26n^2)}{n^5+m^5} + 24 \right) H(r)H(s)^2H(t) \\
& + 5 \left(\frac{nm(n+m)^2(3m^2n+3mn^2-26m^2+34mn-26n^2)}{n^5+m^5} + 24 \right) H(r)^2H(s)H(t) \\
& + \left(\frac{nm(n+m)^2(3m^2n+3mn^2-26m^2+34mn-26n^2)}{n^5+m^5} + 24 \right) H(r)^2H(s)^2 \\
& \left. - 5 \left(\frac{nm(n+m)^2(3m^2n+3mn^2-26m^2+34mn-26n^2)}{n^5+m^5} + 24 \right) H(r)^2H(s)^2H(t) \right) dr \, ds \, dt. \quad (16)
\end{aligned}$$

where $H(t)$ is the distribution of the two data sets under the null hypothesis.

Proof Consider the test statistic

$$A = \frac{nm}{n+m} \int_{-\infty}^{\infty} (\hat{F}(t) - \hat{G}(t))^2 ds.$$

To calculate the third non-centralised moment we need to calculate

$$\begin{aligned} E[T_{n,m}^3] &= \frac{n^3 m^3}{(n+m)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E \left[\left(\hat{F}(s) - \hat{G}(s) \right)^2 \left(\hat{F}(t) - \hat{G}(t) \right)^2 \left(\hat{F}(u) - \hat{G}(u) \right)^2 \right] du dt ds \\ &= \frac{n^3 m^3}{(n+m)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E \left[\right. \end{aligned} \quad (17)$$

$$\frac{1}{n^6} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^n 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (18)$$

$$- \frac{2}{n^5 m} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (19)$$

$$+ \frac{1}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (20)$$

$$- \frac{2}{n^5 m} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (21)$$

$$+ \frac{4}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (22)$$

$$- \frac{2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (23)$$

$$+ \frac{1}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (24)$$

$$- \frac{2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (25)$$

$$+ \frac{1}{n^2 m^4} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (26)$$

$$- \frac{2}{n^5 m} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^n 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (27)$$

$$+ \frac{4}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (28)$$

$$- \frac{2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (29)$$

$$+ \frac{4}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (30)$$

$$- \frac{8}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (31)$$

$$+ \frac{4}{n^2 m^4} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (32)$$

$$- \frac{2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (33)$$

$$(34)$$

$$+ \frac{4}{n^2 m^4} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (35)$$

$$- \frac{2}{n m^5} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (36)$$

$$+ \frac{1}{n^4 m^2} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^n 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (37)$$

$$- \frac{2}{n^3 m^3} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (38)$$

$$+ \frac{1}{n^2 m^4} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (39)$$

$$- \frac{2}{n^3 m^3} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (40)$$

$$+ \frac{4}{n^2 m^4} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (41)$$

$$- \frac{2}{n m^5} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \quad (42)$$

$$+ \frac{1}{n^2 m^4} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \quad (43)$$

$$- \frac{2}{n m^5} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^m 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \quad (44)$$

$$+ \frac{1}{m^6} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m 1_{\{y_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \Big]. \quad (45)$$

It is important to note that we can split the triple integral into the sum of six triple integrals for the following six cases;

1. $s \geq t \geq u$,
2. $s \geq u \geq t$,
3. $t \geq s \geq u$,
4. $t \geq u \geq s$,
5. $u \geq s \geq t$,
6. $u \geq t \geq s$.

We also know that due to symmetry, all six triple integrals will be equal. Thus we will only consider the first scenario and multiply the result by six. Now, we can calculate the integral for each line in the

above expectation separately. Equation (18) is equivalent to,

$$\begin{aligned}
& \frac{1}{n^6} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^n \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \right] \\
&= \frac{1}{n^6} \left\{ n \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \right. \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + n(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 4n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 4n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 4n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 4n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + 2n(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + n(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq u\}}^2 \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + 2n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_k \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_k \geq u\}}^2 \right] \\
&\quad + 4n(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right]
\end{aligned}$$

$$\begin{aligned}
& + 2n(n-1)(n-2)(n-3)\mathbb{E} \left[1_{\{x_k \geq s\}} 1_{\{x_l \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + 8n(n-1)(n-2)(n-3)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_l \geq u\}} \right] \\
& + 8n(n-1)(n-2)(n-3)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_l \geq u\}} \right] \\
& + 8n(n-1)(n-2)(n-3)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + n(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + n(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + n(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
& + 4n(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 4n(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + 4n(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + n(n-1)(n-2)(n-3)(n-4)(n-5)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{1}{n^6} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t nH(u) + 2n(n-1)H(u)^2 + 2n(n-1)H(t)H(u) + 2n(n-1)H(s)H(u) \\
& + n(n-1)H(t)H(u) + n(n-1)H(t)H(u) + n(n-1)H(s)H(u) + 4n(n-1)H(u)^2 \\
& + 4n(n-1)H(u)^2 + 4n(n-1)H(t)H(u) + 4n(n-1)H(u)^2 + 2n(n-1)H(t)H(u) \\
& + 2n(n-1)H(t)H(u) + 2n(n-1)H(u)^2 + n(n-1)(n-2)H(t)H(u)^2 \\
& + n(n-1)(n-2)H(t)^2H(u) + n(n-1)(n-2)H(s)^2H(u) \\
& + 4n(n-1)(n-2)H(t)H(u)^2 + 4n(n-1)(n-2)H(s)H(u)^2 \\
& + 4n(n-1)(n-2)H(s)H(t)H(u) + 2n(n-1)(n-2)H(t)^2H(u) \\
& + 4n(n-1)(n-2)H(t)H(u)^2 + 2n(n-1)(n-2)H(t)H(u)^2 \\
& + 4n(n-1)(n-2)H(t)H(u)^2 + 2n(n-1)(n-2)H(s)H(t)H(u) \\
& + 4n(n-1)(n-2)H(t)H(u)^2 + 2n(n-1)(n-2)H(s)H(u)^2 \\
& + 4n(n-1)(n-2)H(s)H(u)^2 + 2n(n-1)(n-2)H(s)H(t)H(u) \\
& + 4n(n-1)(n-2)H(t)^2H(u) + 2n(n-1)(n-2)H(s)H(t)H(u) \\
& + 4n(n-1)(n-2)H(s)H(t)H(u) + 8n(n-1)(n-2)H(t)H(u)^2 \\
& + 8n(n-1)(n-2)H(t)H(u)^2 + 8n(n-1)(n-2)H(s)H(u)^2 \\
& + n(n-1)(n-2)H(s)H(t)H(u) + 2n(n-1)(n-2)H(s)H(u)^2 \\
& + 2n(n-1)(n-2)H(t)H(u)^2 + 2n(n-1)(n-2)H(t)^2H(u) \\
& + 8n(n-1)(n-2)H(t)H(u)^2 + 2n(n-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + 2n(n-1)(n-2)(n-3)H(t)^2H(u)^2 + 2n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2n(n-1)(n-2)(n-3)H(s)^2H(u)^2 + 2n(n-1)(n-2)(n-3)H(s)H(t)^2H(u) \\
& + 2n(n-1)(n-2)(n-3)H(s)^2H(t)H(u) + 8n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + n(n-1)(n-2)(n-3)H(s)H(t)^2H(u) \\
& + n(n-1)(n-2)(n-3)H(s)^2H(t)H(u) + 4n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 4n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 4n(n-1)(n-2)(n-3)H(s)H(t)^2H(u) \\
& + 2n(n-1)(n-2)(n-3)H(t)^2H(u)^2 + 2n(n-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + 2n(n-1)(n-2)(n-3)H(s)^2H(u)^2 + 8n(n-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + 8n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 8n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + n(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + n(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + n(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u) \\
& + 4n(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 4n(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 4n(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + n(n-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du dt ds.
\end{aligned}$$

Thus the integral of equation (18) is equivalent to

$$\begin{aligned}
& \frac{1}{n^6} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t nH(u) + 16n(n-1)H(u)^2 + 12n(n-1)H(t)H(u) \\
& + 3n(n-1)H(s)H(u) + 45n(n-1)(n-2)H(t)H(u)^2 + 9n(n-1)(n-2)H(t)^2H(u) \\
& + n(n-1)(n-2)H(s)^2H(u) + 20n(n-1)(n-2)H(s)H(u)^2 + 15n(n-1)(n-2)H(s)H(t)H(u) \\
& + 16n(n-1)(n-2)(n-3)H(t)^2H(u)^2 + 35n(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 4n(n-1)(n-2)(n-3)H(s)^2H(u)^2 + 7n(n-1)(n-2)(n-3)H(s)H(t)^2H(u) \\
& + 3n(n-1)(n-2)(n-3)H(s)^2H(t)H(u) + 9n(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 5n(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + n(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u) \\
& + n(n-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

[illegible]

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)\mathbb{E} \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)\mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mH(u)^2 + m(n-1)H(u)^2 + m(n-1)H(t)H(u)^2 + 2m(n-1)H(t)H(u)^2 \\
& + 2m(n-1)H(s)H(u)^2 + m(n-1)H(t)H(u)^2 + m(n-1)H(t)H(u)^2 + m(n-1)H(s)H(u)^2 \\
& + 2m(n-1)H(t)H(u)^2 + 2m(n-1)H(t)H(u)^2 + 2m(n-1)H(s)H(u)^2 + 2m(n-1)H(t)H(u)^2 \\
& + 4m(n-1)H(t)H(u)^2 + 2m(n-1)H(t)H(u)^2 + 2m(n-1)H(t)H(u)^2 + 2m(n-1)H(t)H(u)^2 \\
& + 2m(n-1)H(t)H(u)^2 + m(n-1)H(t)H(u)^2 + m(n-1)H(s)H(u)^2 + m(n-1)H(t)H(u)^2 \\
& + m(n-1)(n-2)H(t)^2H(u)^2 + m(n-1)(n-2)H(s)^2H(u)^2 \\
& + 2m(n-1)(n-2)H(t)H(u)^2 + 2m(n-1)(n-2)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)H(s)H(u)^2 + 2m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)H(t)H(u)^2 + 2m(n-1)(n-2)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)H(t)H(u)^2 + m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 2m(n-1)(n-2)H(t)^2H(u)^2 + 2m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 2m(n-1)(n-2)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)H(t)H(u)^2 \\
& + 2m(n-1)(n-2)H(s)H(t)H(u)^2 + m(n-1)(n-2)H(s)H(u)^2 \\
& + m(n-1)(n-2)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)H(s)^2H(u)^2 \\
& + 2m(n-1)(n-2)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)H(t)^2H(u)^2 + 2m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)H(s)H(t)H(u)^2 + 4m(n-1)(n-2)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)H(t)^2H(u)^2 + 4m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)H(s)H(t)H(u)^2 + m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + m(n-1)(n-2)H(s)H(t)H(u)^2 + m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + m(n-1)(n-2)H(s)H(t)H(u)^2 + m(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 2m(n-1)(n-2)H(t)^2H(u)^2 + 4m(n-1)(n-2)H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(t)^2H(u)^2 + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + m(n-1)(n-2)(n-3)H(s)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(n-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(t)^2H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Thus the integral of equation (19) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + 12mn(n-1)H(t)H(u)^2 \\
& + 3mn(n-1)H(s)H(u)^2 + mn(n-1)(n-2)H(s)^2H(u)^2 + 9mn(n-1)(n-2)H(t)^2H(u)^2 \\
& + 15mn(n-1)(n-2)H(s)H(t)H(u)^2 + 7mn(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 3mn(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + mn(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (20) is equivalent to,

$$\begin{aligned}
& \frac{1}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \right] \\
&= \frac{1}{n^4 m^2} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \right. \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 4m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 4m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 4m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 4m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 4m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + 4m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + 4m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_k \geq u\}}^2 \right] \\
&\quad + 4m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + 2m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_k \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 4m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_k \geq u\}}^2 \right] \\
&\quad + 4m(m-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right]
\end{aligned}$$

$$\begin{aligned}
& + 2m(m-1)(n-2)(n-3)E \left[1_{\{x_k \geq s\}} 1_{\{x_l \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + 8m(m-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_l \geq u\}} \right] \\
& + 8m(m-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_l \geq u\}} \right] \\
& + 8m(m-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + m(m-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + m(m-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
& + 4m(m-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 4m(m-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 4m(m-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + m(m-1)(n-2)(n-3)(n-4)(n-5)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{1}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(t) H(u) + 2m(m-1) H(t) H(u)^2 + 2m(n-1) H(t)^2 H(u) \\
& + 2m(n-1) H(s) H(t) H(u) + m(n-1) H(t) H(u) + m(n-1) H(s) H(t) H(u) \\
& + m(n-1) H(s) H(t) H(u) + 4m(m-1) H(t)^2 H(u)^2 + 4m(m-1) H(s) H(t) H(u)^2 \\
& + 4m(n-1) H(t)^2 H(u) + 4m(m-1) H(t)^2 H(u)^2 + 2m(n-1) H(t)^2 H(u) \\
& + 2m(n-1) H(s) H(t) H(u) + 2m(m-1) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + 4m(m-1)(n-2) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(t)^2 H(u) \\
& + 4m(m-1)(n-2) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t) H(u) + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + 4m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 4m(n-1)(n-2) H(s) H(t)^2 H(u) + 8m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + 8m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 8m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t) H(u) + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u) \\
& + 8m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2)(n-3) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2)(n-3) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + 2m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + 8m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2)(n-3) H(s) H(t) H(u)^2 + m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u) + 4m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 4m(m-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 4m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) \\
& + 2m(m-1)(n-2)(n-3) H(t)^2 H(u)^2 + 2m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u)^2 + 8m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 8m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + 8m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4) H(s)^2 H(t) H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4) H(s)^2 H(t)^2 H(u) \\
& + 4m(m-1)(n-2)(n-3)(n-4) H(s) H(t)^2 H(u)^2 \\
& + 4m(m-1)(n-2)(n-3)(n-4) H(s)^2 H(t)^2 H(u)^2 \\
& + 4m(m-1)(n-2)(n-3)(n-4) H(s)^2 H(t)^2 H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)(n-5) H(s)^2 H(t)^2 H(u)^2 du dt ds.
\end{aligned}$$

Thus the integral of equation (20) is equivalent to

$$\begin{aligned}
& \frac{1}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + mn(m-1)H(t)H(u)^2 + 4mn(n-1)H(t)^2H(u) \\
& + 3mn(n-1)H(s)H(t)H(u) + 4mn(m-1)(n-1)H(t)^2H(u)^2 \\
& + 3mn(m-1)(n-1)H(s)H(t)H(u)^2 + 5mn(n-1)(n-2)H(s)H(t)^2H(u) \\
& + mn(n-1)(n-2)H(s)^2H(t)H(u) + 5mn(m-1)(n-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + mn(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u) \\
& + mn(m-1)(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (21) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^5 m} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \right] \\
&= \frac{1}{n^6} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \right. \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 4m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right]
\end{aligned}$$

[illegible]

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_l \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_l \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}}^2 1_{\{x_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_o \geq t\}} 1_{\{y_p \geq t\}} 1_{\{x_k \geq u\}} 1_{\{x_l \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(t) H(u) + m(n-1) H(t) H(u) + m(n-1) H(t)^2 H(u) + 2m(n-1) H(t) H(u)^2 \\
& + 2m(n-1) H(s) H(t) + m(n-1) H(t)^2 H(u) + m(n-1) H(t)^2 H(u) \\
& + m(n-1) H(s) H(t) H(u) + 2m(n-1) H(t) H(u)^2 + 2m(n-1) H(t) H(u)^2 \\
& + 2m(n-1) H(s) H(t) H(u) + 2m(n-1) H(t)^2 H(u) + 4m(n-1) H(t) H(u)^2 \\
& + 2m(n-1) H(t) H(u)^2 + 2m(n-1) H(t) H(u)^2 + 2m(n-1) H(t) H(u)^2 \\
& + 2m(n-1) H(t)^2 H(u) + m(n-1) H(t)^2 H(u) + m(n-1) H(s) H(t) H(u) \\
& + m(n-1)(n-2) H(t)^2 H(u) + m(n-1)(n-2) H(t)^2 H(u)^2 + m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + 2m(n-1)(n-2) H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t) H(u) + 2m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2) H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + m(n-1)(n-2) H(t)^2 H(u) + m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 2m(n-1)(n-2) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(t)^2 H(u) \\
& + 2m(n-1)(n-2) H(s) H(t)^2 H(u) + m(n-1)(n-2) H(s) H(t) H(u) \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + 2m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 4m(n-1)(n-2) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + 4m(n-1)(n-2) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(t)^2 H(u)^2 + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(t)^2 H(u)^2 + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 2m(n-1)(n-2) H(t)^2 H(u)^2 + 4m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + m(n-1)(n-2)(n-3) H(t)^2 H(u)^2 + m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) + m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u) \\
& + m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 4m(n-1)(n-2)(n-3) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + 2m(n-1)(n-2)(n-3) H(s) H(t) H(u)^2 + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) + 2m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + 4m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2)(n-3) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 \\
& + 4m(n-1)(n-2)(n-3) H(t)^2 H(u)^2 + 4m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 4m(n-1)(n-2)(n-3) H(s) H(t) H(u)^2 + 4m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u) \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds
\end{aligned}$$

Thus the integral of equation (21) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + 4mn(n-1)H(t)^2H(u) + 8mn(n-1)H(t)H(u)^2 \\
& + 3mn(n-1)H(s)H(t)H(u) + 9mn(n-1)(n-2)H(t)^2H(u)^2 \\
& + mn(n-1)(n-2)H(s)^2H(t)H(u) + 5mn(n-1)(n-2)H(s)H(t)^2H(u) \\
& + 10mn(n-1)(n-2)H(s)H(t)H(u)^2 + 7mn(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + mn(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u) + 2mn(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + mn(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{4}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(u)^2 + m(m-1) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 + m(m-1) H(t) H(u)^2 \\
& + m(n-1) H(t) H(u)^2 + 2m(n-1) H(s) H(u)^2 + m(m-1) H(t)^2 H(u)^2 \\
& + m(m-1) H(t)^2 H(u)^2 + 4m(n-1) H(s) H(u)^2 + m(n-1) H(u)^2 \\
& + m(m-1) H(t)^2 H(u)^2 + m(m-1) H(t)^2 H(u)^2 + m(n-1) H(s) H(u)^2 \\
& + 2m(m-1) H(s) H(t) H(u)^2 + 2m(n-1) H(t) H(u)^2 + 2m(m-1) H(s) H(t) H(u)^2 \\
& + 2m(n-1) H(t) H(u)^2 + m(n-1) H(s) H(u)^2 + m(m-1) H(t)^2 H(u)^2 \\
& + m(m-1) H(t)^2 H(u)^2 + m(n-1) H(s) H(u)^2 + m(n-1) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + 2m(m-1) H(t)^2 H(u)^2 + m(n-1) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(n-1)(n-2) H(s)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t) H(u)^2 + m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(n-1)(n-2) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(s)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + m(n-1)(n-2) H(s) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + 2m(m-1)(n-2)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du dt ds
\end{aligned}$$

Thus the integral of equation (22) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + mn(m+4n-5)H(t)H(u)^2 + 3mn(n-1)H(s)H(u)^2 \\
& + 4mn(m-1)(n-1)H(t)^2H(u)^2 + mn(n-1)(3m+5n-13)H(s)H(t)H(u)^2 \\
& + mn(n-1)(n-2)H(s)^2H(u)^2 + 5mn(m-1)(n-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + mn(n-1)(n-2)(n+m-4)H(s)^2H(t)H(u)^2 \\
& + mn(m-1)(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 du dt ds.
\end{aligned}$$

Equation (23) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^m \sum_{p=1}^m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \right] \\
&= \frac{1}{n^6} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \right. \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 4m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_j \geq t\}} 1_{\{y_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_l \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_l \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}}^2 1_{\{x_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_o \geq t\}} 1_{\{y_p \geq t\}} 1_{\{y_k \geq u\}} 1_{\{y_l \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{4}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(t) H(u) + 2m(m-1) H(t) H(u)^2 + m(m-1) H(t)^2 H(u) \\
& + m(n-1) H(s) H(t) H(u) + 2m(n-1) H(s) H(t) H(u) + m(m-1) H(t)^2 H(u) \\
& + m(m-1) H(s) H(t)^2 H(u) + m(n-1) H(s) H(t) H(u) + 2m(m-1) H(t) H(u)^2 \\
& + 2m(m-1) H(s) H(t) H(u)^2 + 4m(m-1) H(s) H(t) H(u)^2 + 2m(m-1) H(s) H(t)^2 H(u) \\
& + 2m(n-1) H(s) H(t) H(u) + 2m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(t) H(u) \\
& + m(m-1) H(s) H(t)^2 H(u) + 2m(m-1) H(s) H(t)^2 H(u) + 2m(m-1) H(s) H(t) H(u)^2 \\
& + m(m-1)(m-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2) H(s)^2 H(t) H(u) + 2m(m-1)(m-2) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 4m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + m(m-1)(n-2) H(t)^2 H(u) + m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + 2m(m-1)(n-2) H(t) H(u)^2 + 2m(m-1)(m-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + 2m(m-1)(m-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 4m(m-1)(m-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(m-1)(n-2) H(s)^2 H(t)^2 H(u) \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(n-2) H(s)^2 H(t)^2 H(u) \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u) + m(n-1)(n-2) H(s) H(t) H(u) \\
& + 2m(m-1)(n-2) H(s)^2 H(t)^2 H(u) + 2m(n-1)(n-2) H(s) H(t) H(u) \\
& + 4m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + 4m(m-1)(n-2) H(s) H(t) H(u)^2 + 4m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(m-2) H(s)^2 H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + 4m(m-1)(n-2) H(s) H(t) H(u)^2 + 4m(m-1)(m-2) H(s)^2 H(t)^2 H(u)^2 \\
& + m(m-1)(m-2)(n-3) H(t)^2 H(u)^2 + m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 2m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u) \\
& + 4m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u) \\
& + m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + 2m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s) H(t) H(u)^2 + 4m(m-1)(m-2)(n-3) H(s)^2 H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u) + 2m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + 2m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(m-1)(m-2)(n-3) H(s)^2 H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 4m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 4m(m-1)(m-2)(n-3) H(s)^2 H(t)^2 H(u)^2 + 4m(m-1)(n-2)(n-3) H(s) H(t) H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 4m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u) \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Thus the integral of equation (23) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + 2mn(m-1)H(t)H(u)^2 + mn(m-1)H(t)^2H(u) \\
& + 3mn(n-1)H(s)H(t)H(u) + 3mn(m-1)(n-1)H(s)H(t)^2H(u) \\
& + 6mn(m-1)(n-1)H(s)H(t)H(u)^2 + mn(m-1)(m-2)H(t)^2H(u)^2 \\
& + mn(n-1)(n-2)H(s)^2H(t)H(u) + 3mn(m-1)(m-2)(n-1)H(s)H(t)^2H(u)^2 \\
& + 2mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 + mn(m-1)(n-1)(n-2)H(s)^2H(t)^2H(u) \\
& + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

The integral of equation (24) is equivalent to the integral of equation (20) but with t and u permuted. Thus the integral of equation (24) is equivalent to

$$\begin{aligned}
& \frac{1}{n^4m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + mn(n-1)H(t)^2H(u) + 4mn(n-1)H(t)H(u)^2 \\
& + 3mn(n-1)H(s)H(t)H(u) + 4mn(m-1)(n-1)H(t)^2H(u)^2 \\
& + 3mn(m-1)(n-1)H(s)H(t)^2H(u) + 5mn(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + mn(n-1)(n-2)H(s)^2H(t)H(u) + 5mn(m-1)(n-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + mn(m-1)(n-1)(n-2)H(s)^2H(t)^2H(u) + mn(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + mn(m-1)(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (25) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \right] \\
&= \frac{1}{n^6} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \right. \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_j \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}}^2 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 4m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_j \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_j \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_j \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}}^2 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_j \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_j \geq t\}} 1_{\{y_l \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}}^2 1_{\{y_j \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_i \geq t\}}^2 1_{\{x_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_j \geq s\}} 1_{\{x_k \geq s\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{y_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_l \geq t\}} 1_{\{y_o \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{x_o \geq s\}} 1_{\{y_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)E \left[1_{\{x_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{-2}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(u)^2 + 2m(m-1)H(t)H(u)^2 + m(m-1)H(t)H(u)^2 \\
& + m(n-1)H(s)H(u)^2 + 2m(n-1)H(s)H(u)^2 + m(m-1)H(t)H(u)^2 \\
& + m(m-1)H(s)H(t)H(u)^2 + m(n-1)H(s)H(u)^2 + 2m(m-1)H(t)H(u)^2 \\
& + 2m(m-1)H(s)H(t)H(u)^2 + 4m(m-1)H(s)H(t)H(u)^2 + 2m(m-1)H(s)H(t)H(u)^2 \\
& + 2m(n-1)H(s)H(u)^2 + 4m(m-1)H(s)H(t)H(u)^2 + m(n-1)H(u)^2 \\
& + m(m-1)H(s)H(t)H(u)^2 + 2m(m-1)H(s)H(t)H(u)^2 + 2m(m-1)H(s)H(t)H(u)^2 \\
& + m(m-1)(m-2)H(t)^2 H(u)^2 + m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + m(n-1)(n-2)H(s)^2 H(u)^2 + 2m(m-1)(m-2)H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)H(u)^2 + 4m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)H(u)^2 + 2m(n-1)(n-2)H(s)^2 H(u)^2 \\
& + m(m-1)(n-2)H(t)H(u)^2 + m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)H(t)H(u)^2 + 2m(m-1)(m-2)H(s)H(t)^2 H(u)^2 \\
& + 2m(m-1)(m-2)H(s)H(t)^2 H(u)^2 + 2m(m-1)(m-2)H(s)H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)H(u)^2 + 4m(m-1)(m-2)H(s)H(u)^2 H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)H(u)^2 + 2m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(m-1)(n-2)H(s)^2 H(t)H(u)^2 + 2m(m-1)(n-2)H(s)^2 H(t)H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)H(u)^2 + 2m(m-1)(n-2)H(s)^2 H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)H(u)^2 + m(n-1)(n-2)H(s)H(u)^2 \\
& + 2m(m-1)(n-2)H(s)^2 H(t)H(u)^2 + 2m(n-1)(n-2)H(s)H(u)^2 \\
& + 4m(m-1)(m-2)H(s)H(t)^2 H(u)^2 + 4m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(m-1)(m-2)H(s)H(t)^2 H(u)^2 + 4m(m-1)(n-2)H(s)^2 H(t)H(u)^2 \\
& + 4m(m-1)(n-2)H(s)H(t)H(u)^2 + 4m(m-1)(n-2)H(s)^2 H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)H(u)^2 + 2mm-1)(n-2)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(m-2)H(s)^2 H(t)^2 H(u)^2 + 2m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + 4m(m-1)(n-2)H(s)H(t)H(u)^2 + 4m(m-1)(m-2)H(s)^2 H(t)^2 H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(t)^2 H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 + 2m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 + m(n-1)(n-2)(n-3)H(s)^2 H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 + 4m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 + 2m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 4m(m-1)(m-2)(n-3)H(s)^2 H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 + 2m(m-1)(m-2)(n-3)H(s)^2 H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2 H(t)H(u)^2 + 4m(m-1)(m-2)(n-3)H(s)H(t)^2 H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)H(s)^2 H(t)^2 H(u)^2 + 4m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 4m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Thus the integral of equation (25) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + 3mn(m-1)H(t)H(u)^2 + 3mn(n-1)H(s)H(u)^2 \\
& + 9mn(m-1)(n-1)H(s)H(t)H(u)^2 + mn(m-1)(m-2)H(t)^2H(u)^2 \\
& + mn(n-1)(n-2)H(s)^2H(t)^2 + 3mn(m-1)(m-2)(n-1)H(s)H(t)^2H(u)^2 \\
& + 3mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

The integral of equation (26) is equivalent to the integral of equation (24) but with s and t permuted and n and m permuted. Thus the integral of equation (24) is equivalent to

$$\begin{aligned}
& \frac{1}{n^2m^4} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(s)H(u) + mn(m-1)H(s)^2H(u) + 4mn(m-1)H(s)H(u)^2 \\
& + 3mn(m-1)H(s)H(t)H(u) + 4mn(m-1)(n-1)H(s)^2H(u)^2 \\
& + 3mn(m-1)(n-1)H(s)^2H(t)H(u) + 5mn(m-1)(m-2)H(s)H(t)H(u)^2 \\
& + mn(m-1)(m-2)H(s)H(t)^2H(u) + 5mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\
& + mn(m-1)(m-2)(n-1)H(s)^2H(t)^2H(u) + mn(m-1)(m-2)(m-3)H(s)H(t)^2H(u)^2 \\
& + mn(m-1)(m-2)(m-3)(n-1)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (27) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^5 m} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^n \mathbb{E} \left[1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq s\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \right] \\
&= \frac{1}{n^6} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \right. \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 4m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_l \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_j \geq u\}} 1_{\{x_l \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_l \geq u\}} 1_{\{x_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)E \left[1_{\{x_o \geq s\}} 1_{\{y_p \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq u\}} 1_{\{x_l \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(s) H(u) + m(n-1) H(s) H(u) + m(n-1) H(s)^2 H(u) \\
& + 2m(n-1) H(s) H(u)^2 + 2m(n-1) H(s) H(t) H(u) + m(n-1) H(s)^2 H(u) \\
& + m(n-1) H(s) H(t) H(u) + 2m(n-1) H(s) H(u)^2 + 2m(n-1) H(s) H(u)^2 \\
& + 2m(n-1) H(s) H(t) H(u) + 2m(n-1) H(s) H(t) H(u) + 4m(n-1) H(s) H(u)^2 \\
& + 2m(n-1) H(s) H(u)^2 + 2m(n-1) H(s) H(u)^2 + 2m(n-1) H(s) H(u)^2 \\
& + 2m(n-1) H(s) H(t) H(u) + m(n-1) H(s) H(t) H(u) + m(n-1) H(s) H(t) H(u) \\
& + m(n-1)(n-2) H(s)^2 H(u) + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s) H(u)^2 \\
& + 2m(n-1)(n-2) H(s)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u) \\
& + 2m(n-1)(n-2) H(s)^2 H(t) H(u) + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(u)^2 \\
& + 2m(n-1)(n-2) H(s)^2 H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u) \\
& + m(n-1)(n-2) H(s)^2 H(t) H(u) + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + 2m(n-1)(n-2) H(s) H(t) H(u) + 2m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + m(n-1)(n-2) H(s) H(t) H(u) + m(n-1)(n-2) H(s)^2 H(t) H(u) \\
& + 2m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 2m(n-1)(n-2) H(s) H(t)^2 H(u) + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t) H(u)^2 + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(u)^2 + 4m(n-1)(n-2) H(s)^2 H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s)^2 H(t) H(u) + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s)^2 H(u)^2 \\
& + 4m(n-1)(n-2) H(s) H(t) H(u)^2 + 4m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2)(n-3) H(s)^2 H(u)^2 + m(n-1)(n-2)(n-3) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u) \\
& + m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) + m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 4m(n-1)(n-2)(n-3) H(s) H(t) H(u)^2 + 4m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 \\
& + m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u) + m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 \\
& + m(n-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + 2m(n-1)(n-2)(n-3) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2)(n-3) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u)
\end{aligned}$$

$$\begin{aligned}
& + 2m(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u) + 4m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)^2H(u)^2 + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u) \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 4m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Thus the integral of equation (27) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^5 m} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(s)H(u) + mn(n-1)H(s)^2H(u) + 8mn(n-1)H(s)H(u)^2 \\
& + 6mn(n-1)H(s)H(t)H(u) + 15mn(n-1)(n-2)H(s)H(t)H(u)^2 \\
& + 3mn(n-1)(n-2)H(s)H(t)^2H(u) + 3mn(n-1)(n-2)H(s)^2H(t)H(u) \\
& + 4mn(n-1)(n-2)H(t)^2H(u)^2 + 5mn(n-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + mn(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u) + 4mn(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + mn(n-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (28) is equivalent to

$$\begin{aligned}
& \frac{4}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^n \sum_{p=1}^m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \right] \\
&= \frac{4}{n^4 m^2} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \right. \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_i \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_j \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{x_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{x_j \geq u\}} \right] \left. \right\}
\end{aligned}$$

[illegible]

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{4}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(u)^2 + m(m-1) H(s) H(u)^2 + m(m-1) H(s) H(u)^2 \\
& + m(n-1) H(t) H(u)^2 + 2m(n-1) H(t) H(u)^2 + m(m-1) H(s)^2 H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 + m(n-1) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(m-1) H(s)^2 H(u)^2 + m(n-1) H(t) H(u)^2 \\
& + 2m(m-1) H(s) H(t) H(u)^2 + 2m(n-1) H(t) H(u)^2 + 2m(m-1) H(s) H(t) H(u)^2 \\
& + 2m(n-1) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 + m(m-1) H(s) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 + m(n-1) H(s) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + 2m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s)^2 H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s)^2 H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + 2m(n-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 + m(n-1)(n-2) H(t) H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + m(m-1)(n-2)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)^2H(t)H(u)^2 + 2m(n-1)(n-2)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Thus the integral of equation (28) is equivalent to

$$\begin{aligned}
& \frac{4}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + mn(n+m-2)H(s)H(u)^2 + 6mn(n-1)H(t)H(u)^2 \\
& + mn(m-1)(n-1)H(s)^2H(u)^2 + 3mn(n-1)(2m+n-4)H(s)H(t)H(u)^2 \\
& + 3mn(n-1)(n-2)H(t)^2H(u)^2 + 3mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + mn(n-1)(n-2)(3m+n-6)H(s)H(t)^2H(u)^2 \\
& + mn(m-1)(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (29) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3 m^3} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^n \sum_{o=1}^m \sum_{p=1}^m \mathbb{E} \left[1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq s\}} 1_{\{y_o \geq u\}} 1_{\{y_p \geq u\}} \right] \\
&= \frac{1}{n^6} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \right. \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 4m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
&\quad + m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1)(n-2) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \left. \right\}
\end{aligned}$$

[illegible]

$$\begin{aligned}
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_l \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)E \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_j \geq u\}} 1_{\{y_l \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_i \geq t\}}^2 1_{\{y_j \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_i \geq u\}}^2 \right] \\
& + m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 4m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_k \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_l \geq u\}} 1_{\{y_o \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{x_i \geq s\}} 1_{\{y_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + 2m(n-1)(n-2)(n-3)(n-4)E \left[1_{\{y_i \geq s\}} 1_{\{x_k \geq s\}} 1_{\{x_l \geq t\}} 1_{\{x_o \geq t\}} 1_{\{y_i \geq u\}} 1_{\{y_j \geq u\}} \right] \\
& + m(n-1)(n-2)(n-3)(n-4)(n-5)E \left[1_{\{x_o \geq s\}} 1_{\{y_p \geq s\}} 1_{\{x_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_k \geq u\}} 1_{\{y_l \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{-2}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(t) H(u) + 2m(m-1) H(t) H(u)^2 + m(m-1) H(s) H(t) H(u) \\
& + m(n-1) H(s) H(t) H(u) + 2m(n-1) H(t)^2 H(u) + m(m-1) H(s) H(t) H(u) \\
& + m(m-1) H(s)^2 H(t) H(u) + m(n-1) H(s) H(t) H(u) + 2m(m-1) H(t) H(u)^2 \\
& + 2m(m-1) H(s) H(t) H(u)^2 + 4m(m-1) H(t)^2 H(u)^2 + 2m(m-1) H(s) H(t)^2 H(u) \\
& + 2m(n-1) H(t)^2 H(u) + 4m(m-1) H(t)^2 H(u)^2 + m(n-1) H(t) H(u) \\
& + m(m-1) H(s)^2 H(t) H(u) + 2m(m-1) H(s) H(t)^2 H(u) + 2m(m-1) H(s) H(t) H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u) \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(m-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 4m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(m-1)(n-2) H(s) H(t) H(u) + m(m-1)(n-2) H(s)^2 H(t) H(u) \\
& + 2m(m-1)(n-2) H(t) H(u)^2 + 2m(m-1)(m-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(m-2) H(s)^2 H(t) H(u)^2 + 2m(m-1)(m-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 4m(m-1)(m-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 4m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s)^2 H(t)^2 H(u) \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(n-2) H(s)^2 H(t)^2 H(u) \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u) + m(n-1)(n-2) H(s) H(t) H(u) \\
& + 2m(m-1)(n-2) H(s)^2 H(t)^2 H(u) + 2m(n-1)(n-2) H(t)^2 H(u) \\
& + 4m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + 4m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 4m(m-1)(n-2) H(t)^2 H(u)^2 + 4m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u) + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(m-2) H(s)^2 H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + 4m(m-1)(n-2) H(t)^2 H(u)^2 + 4m(m-1)(m-2) H(s)^2 H(t)^2 H(u)^2 \\
& + m(m-1)(m-2)(n-3) H(s) H(t) H(u)^2 + m(m-1)(m-2)(n-3) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(m-2)(n-3) H(s)^2 H(t) H(u)^2 + 2m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) \\
& + m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + m(n-1)(n-2)(n-3) H(s) H(t)^2 H(u) \\
& + 4m(m-1)(m-2)(n-3) H(s) H(t)^2 H(u)^2 + 4m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2)(n-3) H(s)^2 H(t) H(u)^2 + m(m-1)(n-2)(n-3) H(s)^2 H(t) H(u) \\
& + m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u) + 2m(m-1)(m-2)(n-3) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s) H(t) H(u)^2 + 4m(m-1)(m-2)(n-3) H(s)^2 H(t)^2 H(u)^2 \\
& + 2m(m-1)(n-2)(n-3) H(s) H(t)^2 H(u) + 2m(m-1)(n-2)(n-3) H(s)^2 H(t)^2 H(u)
\end{aligned}$$

$$\begin{aligned}
& + 2m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 4m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + 4m(m-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + 4m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 4m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u) \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 4m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Thus the integral of equation (29) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + 2mn(m-1)H(t)H(u)^2 + 2mn(n-1)H(t)^2H(u) \\
& + mn(m+n-2)H(s)H(t)H(u) + mn(m-1)(n-1)H(s)^2H(t)H(u) \\
& + mn(m-1)(2n+m-4)H(s)H(t)H(u)^2 + mn(n-1)(2m+n-4)H(s)H(t)^2H(u) \\
& + 4mn(m-1)(n-1)H(t)^2H(u)^2 + mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\
& + mn(m-1)(n-1)(n-2)H(s)^2H(t)^2H(u) \\
& + 2mn(m-1)(n-1)(m+n-4)H(s)H(t)^2H(u)^2 \\
& + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

Equation (30) is equivalent to

$$\begin{aligned}
& \frac{4}{n^4 m^2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^n \sum_{l=1}^m \sum_{o=1}^n \sum_{p=1}^n \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{x_p \geq u\}} \right] \\
&= \frac{4}{n^4 m^2} \left\{ m \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \right. \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + 2m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_i \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{y_i \geq s\}} 1_{\{x_j \geq s\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + 2m(m-1) \mathbb{E} \left[1_{\{x_i \geq s\}} 1_{\{y_i \geq s\}} 1_{\{x_j \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}} 1_{\{x_j \geq u\}} \right] \\
&\quad + m(n-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \\
&\quad + m(m-1) \mathbb{E} \left[1_{\{x_j \geq s\}} 1_{\{y_j \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{x_i \geq u\}}^2 \right] \left. \right\}
\end{aligned}$$

[illegible]

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{4}{n^4 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(t) H(u) + m(m-1) H(s) H(t) H(u) + m(n-1) H(u) H(t)^2 \\
& + m(m-1) H(s) H(t) H(u) + m(n-1) H(s) H(t) H(u) + 2m(n-1) H(u)^2 H(t) \\
& + m(m-1) H(s) H(t)^2 H(u) + m(m-1) H(s)^2 H(t) H(u) + m(n-1) H(u) H(t)^2 \\
& + m(n-1) H(t) H(u) + m(m-1) H(s)^2 H(t) H(u) + m(m-1) H(s) H(t)^2 H(u) \\
& + m(n-1) H(u) H(t)^2 + 2m(m-1) H(s) H(t) H(u)^2 + 2m(n-1) H(u)^2 H(t) \\
& + 2m(m-1) H(s) H(t) H(u)^2 + 2m(n-1) H(u)^2 H(t) + m(n-1) H(u)^2 H(t) \\
& + m(m-1) H(s) H(t) H(u)^2 + m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(u)^2 H(t) \\
& + m(n-1) H(u) H(t)^2 + m(m-1) H(s) H(t)^2 H(u) + 2m(m-1) H(s) H(t) H(u)^2 \\
& + m(n-1) H(s) H(t) H(u) + m(m-1) H(s) H(t)^2 H(u) + m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u) + m(n-1)(n-2) H(u)^2 H(t)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u) + m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u) + m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(u)^2 H(t)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u) + m(m-1)(n-2) H(u) H(t)^2 H(s)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u) + m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2) H(u) H(t)^2 + m(m-1)(n-2) H(u) H(t)^2 H(s)^2 \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u) + m(m-1)(n-2) H(u) H(t)^2 H(s)^2 \\
& + m(m-1)(n-2) H(s)^2 H(t) H(u) + m(n-1)(n-2) H(s) H(t) H(u) \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u) + m(m-1)(n-2) H(u) H(t)^2 H(s)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u) \\
& + m(n-1)(n-2) H(s) H(t)^2 H(u) + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + 2m(n-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t)^2 H(u) + m(n-1)(n-2) H(s) H(t)^2 H(u) \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(u)^2 H(t)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(u)^2 H(t)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t) H(u)^2 + 2m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + 2m(n-1)(n-2) H(u)^2 H(t) + 2m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + 2m(m-1)(n-2) H(s) H(t)^2 H(u)^2 + 2m(n-1)(n-2) H(u)^2 H(t)^2 \\
& + m(m-1)(n-2) H(u) H(t)^2 H(s)^2 + m(n-1)(n-2) H(u) H(t)^2
\end{aligned}$$

$$\begin{aligned}
& + m(m-1)(n-2)H(u)H(t)^2H(s)^2 + m(m-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)^2H(u)^2 + 2m(n-1)(n-2)H(u)^2H(t) \\
& + 2m(m-1)(n-2)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u) \\
& + m(m-1)(n-2)(n-3)H(u)H(t)^2H(s)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u) \\
& + m(m-1)(n-2)(n-3)H(u)H(t)^2H(s)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(u)H(t)^2H(s)^2 \\
& + m(m-1)(n-2)(n-3)H(u)H(t)^2H(s)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u) + m(m-1)(n-2)(n-3)H(u)H(t)^2H(s)^2 \\
& + m(m-1)(n-2)(n-3)H(u)H(t)^2H(s)^2 + m(n-1)(n-2)(n-3)H(s)H(t)^2H(u) \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(n-1)(n-2)(n-3)H(u)^2H(t)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + 2m(n-1)(n-2)(n-3)H(u)^2H(t)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + 2m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + 2m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(u)H(t)^2H(s)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + 2m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(n-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du dt ds.
\end{aligned}$$

Thus the integral of equation (30) is equivalent to

$$\begin{aligned}
& \frac{-2}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + mn(m+n-2)H(s)H(t)H(u) + 2mn(n-1)H(t)^2H(u) \\
& + 4mn(n-1)H(t)H(u)^2 + mn(n-1)(2m+n-4)H(s)H(t)^2H(u) \\
& + mn(m-1)(n-1)H(s)^2H(t)H(u) + 2mn(n-1)(2m+n-4)H(s)H(t)H(u)^2 \\
& + 3mn(n-1)(n-2)H(t)^2H(u)^2 + mn(m-1)(n-1)(n-2)H(s)^2H(t)^2H(u) \\
& + mn(n-1)(n-2)(3m+n-6)H(s)H(t)^2H(u)^2 + 2mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + mn(m-1)(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 du \, dt \, ds.
\end{aligned}$$

$$\begin{aligned}
& + m(m-1)(m-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{x_i \geq t\}} 1_{\{y_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + m(m-1)(n-2)(n-3)(n-4)E \left[1_{\{x_l \geq s\}} 1_{\{y_o \geq s\}} 1_{\{y_i \geq t\}} 1_{\{x_j \geq t\}} 1_{\{y_i \geq u\}} 1_{\{x_k \geq u\}} \right] \\
& + m(m-1)(m-2)(n-3)(n-4)(n-5)E \left[1_{\{x_i \geq s\}} 1_{\{y_j \geq s\}} 1_{\{x_k \geq t\}} 1_{\{y_l \geq t\}} 1_{\{x_o \geq u\}} 1_{\{y_p \geq u\}} \right] \Bigg\}.
\end{aligned}$$

Each expectation can be calculated and by taking the integral over the first scenario we get

$$\begin{aligned}
& \frac{-8}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t m H(u)^2 + m(m-1) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 \\
& + m(n-1) H(t) H(u)^2 + m(m-1) H(s) H(u)^2 + m(n-1) H(s) H(u)^2 \\
& + m(m-1) H(t)^2 H(u)^2 + m(m-1) H(t)^2 H(u)^2 + m(m-1) H(s)^2 H(u)^2 \\
& + m(m-1) H(s) H(u)^2 + m(m-1) H(t)^2 H(u)^2 + m(m-1) H(t)^2 H(u)^2 \\
& + m(n-1) H(s) H(u)^2 + m(m-1) H(t) H(u)^2 + m(m-1) H(s) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 + m(m-1) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(m-1) H(s) H(t) H(u)^2 + m(n-1) H(t) H(u)^2 \\
& + m(n-1) H(u)^2 + m(m-1) H(t)^2 H(u)^2 + m(m-1) H(t)^2 H(u)^2 \\
& + m(m-1) H(s)^2 H(u)^2 + m(m-1) H(s) H(t) H(u)^2 + m(m-1) H(s) H(t) H(u)^2 \\
& + m(m-1) H(t)^2 H(u)^2 + m(m-1) H(s) H(t) H(u)^2 + m(m-1) H(t) H(u)^2 \\
& + m(m-1) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(n-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s)^2 H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(m-2) H(s)^2 H(t) H(u)^2 + m(m-1)(n-2) H(s) H(t) H(u)^2 \\
& + m(m-1)(n-2) H(s) H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s) H(t)^2 H(u)^2 + m(m-1)(n-2) H(t)^2 H(u)^2 \\
& + m(m-1)(n-2) H(t)^2 H(u)^2 + m(m-1)(n-2) H(s) H(t)^2 H(u)^2 \\
& + m(m-1)(m-2) H(s)^2 H(t) H(u)^2 + m(m-1)(n-2) H(s)^2 H(t) H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + m(m-1)(m-2)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)H(u)^2 + m(m-1)(n-2)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)H(t)H(u)^2 + m(m-1)(m-2)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(t)^2H(u)^2 + m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(s)^2H(t)H(u)^2 + m(n-1)(n-2)H(t)H(u)^2 \\
& + m(m-1)(n-2)H(t)H(u)^2 + m(m-1)(m-2)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)H(t)^2H(u)^2 + m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(s)^2H(t)H(u)^2 + m(n-1)(n-2)H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)H(u)^2 + m(m-1)(n-2)H(s)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(s)^2H(u)^2 + m(n-1)(n-2)H(s)H(u)^2 \\
& + m(m-1)(m-2)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)H(s)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)H(u)^2 + m(m-1)(m-2)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)H(t)^2H(u)^2 + m(m-1)(n-2)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)H(s)H(t)H(u)^2 \\
& + m(m-1)(n-2)H(s)H(t)H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(n-1)(n-2)(n-3)H(s)H(t)H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2
\end{aligned}$$

$$\begin{aligned}
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)H(t)^2H(u)^2 + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)H(s)^2H(t)^2H(u)^2 + m(m-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)H(s)^2H(t)^2H(u)^2 \\
& + m(m-1)(n-2)(n-3)(n-4)H(s)^2H(t)H(u)^2 \\
& + m(m-1)(m-2)(n-3)(n-4)(n-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds
\end{aligned}$$

Thus the integral of equation (31) is equivalent to

$$\begin{aligned} & \frac{-2}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + 2mn(n+m-2)H(t)H(u)^2 + mn(n+m-2)H(s)H(u)^2 \\ & + 4mn(m-1)(n-1)H(t)^2H(u)^2 + mn(m-1)(n-1)H(s)^2H(u)^2 \\ & + mn(m^2 + 4mn + n^2 - 7m - 7n + 8)H(s)H(t)H(u)^2 \\ & + 2mn(m-1)(n-2)(m+n-4)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(n-1)(m+n-4)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (32) is equivalent to the integral of equation (30) but with n and m permuted. Thus the integral of equation (32) is equivalent to

$$\begin{aligned} & \frac{4}{n^2 m^4} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + mn(m+n-2)H(s)H(t)H(u) + 2mn(m-1)H(t)^2H(u) \\ & + 4mn(m-1)H(t)H(u)^2 + mn(m-1)(2n+m-4)H(s)H(t)^2H(u) \\ & + mn(m-1)(n-1)H(s)^2H(t)H(u) + 2mn(m-1)(2n+m-4)H(s)H(t)H(u)^2 \\ & + 3mn(m-1)(m-2)H(t)^2H(u)^2 + mn(m-1)(m-2)(n-1)H(s)^2H(t)^2H(u) \\ & + mn(m-1)(m-2)(3n+m-6)H(s)H(t)^2H(u)^2 \\ & + 2mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(n-1)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (33) is equivalent to the integral of equation (29) but with n and m permuted. Thus the integral of equation (33) is equivalent to

$$\begin{aligned} & \frac{-2}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + 2mn(m-1)H(t)^2H(u) + 2mn(n-1)H(t)H(u)^2 \\ & + mn(m+n-2)H(s)H(t)H(u) + mn(m-1)(n-1)H(s)^2H(t)H(u) \\ & + mn(m-1)(2n+m-4)H(s)H(t)^2H(u) + mn(n-1)(2m+n-4)H(s)H(t)H(u)^2 \\ & + 4mn(m-1)(n-1)H(t)^2H(u)^2 + mn(m-1)(m-2)(n-1)H(s)^2H(t)^2H(u) \\ & + mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 \\ & + 2mn(m-1)(n-1)(n+m-4)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (35) is equivalent to the integral of equation (28) but with n and m permuted. Thus the integral of equation (35) is equivalent to

$$\begin{aligned} & \frac{4}{n^2 m^4} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + mn(m+n-2)H(s)H(u)^2 + 6mn(m-1)H(t)H(u)^2 \\ & + mn(m-1)(n-1)H(s)^2H(u)^2 + 3mn(m-1)(2n+m-4)H(s)H(t)H(u)^2 \\ & + 3mn(m-1)(m-2)H(t)^2H(u)^2 + 3mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(3n+m-6)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(n-1)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (36) is equivalent to the integral of equation (27) but with n and m permuted. Thus the integral of equation (36) is equivalent to

$$\begin{aligned} & \frac{-2}{nm^5} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(s)H(u) + mn(m-1)H(s)^2H(u) + 8mn(m-1)H(s)H(u)^2 \\ & + 6mn(m-1)H(s)H(t)H(u) + 15mn(m-1)(m-2)H(s)H(t)H(u)^2 \\ & + 3mn(m-1)(m-2)H(s)H(t)^2H(u) + 3mn(m-1)(m-2)H(s)^2H(t)H(u) \\ & + 4mn(m-1)(m-2)H(s)^2H(u)^2 + 5mn(m-1)(m-2)(m-3)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)H(s)^2H(t)^2H(u) \\ & + 4mn(m-1)(m-2)(m-3)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(m-4)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (37) is equivalent to the integral of equation (26) but with n and m permuted. Thus the integral of equation (37) is equivalent to

$$\begin{aligned} & \frac{1}{n^4m^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(s)H(u) + mn(m-1)H(s)^2H(u) + 4mn(n-1)H(s)H(u)^2 \\ & + 3mn(n-1)H(s)H(t)H(u) + 4mn(m-1)(n-2)H(s)^2H(u)^2 \\ & + 3mn(m-1)(n-1)H(s)^2H(t)H(u) + 5mn(n-1)(n-2)H(s)H(t)H(u)^2 \\ & + mn(n-1)(n-2)H(s)H(t)^2H(u) + 5mn(m-1)(n-1)(n-2)H(s)^2H(t)^2H(u) \\ & + mn(m-1)(n-1)(n-2)H(s)^2H(t)H(u)^2 + mn(n-1)(n-2)(n-3)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(n-1)(n-2)(n-3)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (38) is equivalent to the integral of equation (25) but with n and m permuted. Thus the integral of equation (38) is equivalent to

$$\begin{aligned} & \frac{-2}{n^3m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + 3mn(n-1)H(t)H(u)^2 + 3mnH(s)H(u)^2 \\ & + 9mn(m-1)(n-1)H(s)H(t)H(u)^2 + mn(n-1)(n-2)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)H(s)^2H(u)^2 + 3mn(m-1)(n-1)(n-2)H(s)H(t)^2H(u)^2 \\ & + 3mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (39) is equivalent to the integral of equation (24) but with n and m permuted. Thus the integral of equation (39) is equivalent to

$$\begin{aligned} & \frac{1}{n^2m^4} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + mn(m-1)H(t)^2H(u) + 4mn(m-1)H(t)H(u)^2 \\ & + 3mn(m-1)H(s)H(t)H(u) + 4mn(m-1)(n-2)H(t)^2H(u)^2 \\ & + 3mn(m-1)(n-1)H(s)H(t)^2H(u) + 5mn(m-1)(m-2)H(s)H(t)H(u)^2 \\ & + mn(m-1)(m-2)H(s)^2H(t)H(u) + 5mn(m-1)(m-2)(n-1)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(n-1)H(s)^2H(t)^2H(u) \\ & + mn(m-1)(m-2)(m-3)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(n-1)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (40) is equivalent to the integral of equation (23) but with n and m permuted. Thus the integral of equation (40) is equivalent to

$$\begin{aligned} & \frac{-2}{n^3 m^3} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + 2mn(n-1)H(t)H(u)^2 + mn(n-1)H(t)^2H(u) \\ & + 3mn(m-1)H(s)H(t)H(u) + 3mn(m-1)(n-1)H(s)H(t)^2H(u) \\ & + 6mn(m-1)(n-1)H(s)H(t)H(u)^2 + mn(n-1)(n-2)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)H(s)^2H(t)H(u) + 3mn(m-1)(n-1)(n-2)H(s)H(t)^2H(u)^2 \\ & + 2mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(n-1)H(s)^2H(t)^2H(u) \\ & + mn(m-1)(m-2)(n-1)(n-2)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (41) is equivalent to the integral of equation (22) but with n and m permuted. Thus the integral of equation (41) is equivalent to

$$\begin{aligned} & \frac{4}{n^2 m^4} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + mn(n+4m-5)H(t)H(u)^2 + 3mn(m-1)H(s)H(u)^2 \\ & + 4mn(m-1)(n-1)H(t)^2H(u)^2 + mn(m-1)(3n+5m-13)H(s)H(t)H(u)^2 \\ & + mn(m-1)(m-2)H(s)^2H(u)^2 + 5mn(m-1)(m-2)(n-1)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(n+m-4)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(n-1)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (42) is equivalent to the integral of equation (21) but with n and m permuted. Thus the integral of equation (42) is equivalent to

$$\begin{aligned} & \frac{-2}{nm^5} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + 4mn(m-1)H(t)^2H(u) + 8mn(m-1)H(t)H(u)^2 \\ & + 3mn(m-1)H(s)H(t)H(u) + 9mn(m-1)(m-2)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)H(s)^2H(t)H(u) + 5mn(m-1)(m-2)H(s)H(t)^2H(u) \\ & + 10mn(m-1)(m-2)H(s)H(t)H(u)^2 + 7mn(m-1)(m-2)(m-3)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(m-3)H(s)^2H(t)^2H(u) \\ & + 2mn(m-1)(m-2)(m-3)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(m-4)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (43) is equivalent to the integral of equation (20) but with n and m permuted. Thus the integral of equation (43) is equivalent to

$$\begin{aligned} & \frac{1}{n^2 m^4} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(t)H(u) + mn(n-1)H(t)H(u)^2 + 4mn(m-1)H(t)^2H(u) \\ & + 3mn(m-1)H(s)H(t)H(u) + 4mn(m-1)(n-1)H(t)^2H(u)^2 \\ & + 3mn(m-1)(n-1)H(s)H(t)H(u)^2 + 5mn(m-1)(m-2)H(s)H(t)^2H(u) \\ & + mn(m-1)(m-2)H(s)^2H(t)H(u) + 5mn(m-1)(m-2)(n-1)H(s)H(t)^2H(u)^2 \\ & + mn(m-1)(m-2)(n-1)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)H(s)^2H(t)^2H(u) \\ & + mn(m-1)(m-2)(m-3)(n-1)H(s)^2H(t)^2H(u)^2 du dt ds. \end{aligned}$$

The integral of equation (44) is equivalent to the integral of equation (19) but with n and m permuted. Thus the integral of equation (44) is equivalent to

$$\begin{aligned} & \frac{-2}{nm^5} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mnH(u)^2 + 12mn(m-1)H(t)H(u)^2 + 3mn(m-1)H(s)H(u)^2 \\ & + mn(m-1)(m-2)H(s)^2H(u)^2 + 9mn(m-1)(m-2)H(t)^2H(u)^2 \\ & + 15mn(m-1)(m-2)H(s)H(t)H(u)^2 + 7mn(m-1)(m-2)(m-3)H(s)H(t)^2H(u)^2 \\ & + 3mn(m-1)(m-2)(m-3)H(s)^2H(t)H(u)^2 \\ & + mn(m-1)(m-2)(m-3)(m-4)H(s)^2H(t)^2H(u)^2 du \, dt \, ds. \end{aligned}$$

The integral of equation (45) is equivalent to the integral of equation (18) but with n and m permuted. Thus the integral of equation (45) is equivalent to

$$\begin{aligned} & \frac{1}{m^6} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t mH(u) + 16m(m-1)H(u)^2 + 12m(m-1)H(t)H(u) \\ & + 3m(m-1)H(s)H(u) + 45m(m-1)(m-2)H(t)H(u)^2 \\ & + 9m(m-1)(m-2)H(t)^2H(u) + m(m-1)(m-2)H(s)^2H(u) \\ & + 20m(m-1)(m-2)H(s)H(u)^2 + 15m(m-1)(m-2)H(s)H(t)H(u) \\ & + 16m(m-1)(m-2)(m-3)H(t)^2H(u)^2 + 35m(m-1)(m-2)(m-3)H(s)H(t)H(u)^2 \\ & + 4m(m-1)(m-2)(m-3)H(s)^2H(u)^2 + 7m(m-1)(m-2)(m-3)H(s)H(t)^2H(u) \\ & + 3m(m-1)(m-2)(m-3)H(s)^2H(t)H(u) \\ & + 9m(m-1)(m-2)(m-3)(m-4)H(s)H(t)^2H(u)^2 \\ & + 5m(m-1)(m-2)(m-3)(m-4)H(s)^2H(t)H(u)^2 \\ & + m(m-1)(m-2)(m-3)(m-4)H(s)^2H(t)^2H(u) \\ & + m(m-1)(m-2)(m-3)(m-4)(m-5)H(s)^2H(t)^2H(u)^2 du \, dt \, ds. \end{aligned}$$

If we collect all the integrals of equation (18) to (45) and multiply the result by six then we get

$$\begin{aligned}
E[T_{n,m}^3] = & \frac{6(n^5 + m^5)}{(n+m)^3(nm)^2} \int_{-\infty}^{\infty} \int_{-\infty}^s \int_{-\infty}^t H(u) \left(1 + 2 \left(\frac{nm(n+m)^2(7m^2 - 10nm + 7n^2)}{n^5 + m^5} - 8 \right) H(u) \right. \\
& + 2 \left(\frac{nm(n+m)^2(5m^2 - 7nm + 5n^2)}{n^5 + m^5} - 6 \right) H(t) + \left(\frac{nm(n+m)^2(m^2 - nm + n^2)}{n^5 + m^5} - 3 \right) H(s) \\
& + 5 \left(\frac{nm(n+m)^2(2m^2n + 2mn^2 - 19m^2 + 25mn - 19n^2)}{n^5 + m^5} + 18 \right) H(t)H(u) \\
& + \left(\frac{nm(n+m)^2(2m^2n + 2mn^2 - 19m^2 + 25mn - 19n^2)}{n^5 + m^5} + 18 \right) H(t)^2 \\
& - \left(\frac{nm(n+m)^2(m^2 - nm + n^2)}{n^5 + m^5} - 2 \right) H(s)^2 \\
& + 2 \left(\frac{nm(n+m)^2(m^2n + mn^2 - 19m^2 + 26mn - 19n^2)}{n^5 + m^5} + 20 \right) H(s)H(u) \\
& + \left(\frac{nm(n+m)^2(m^2n + mn^2 - 27m^2 + 37mn - 27n^2)}{n^5 + m^5} + 30 \right) H(s)H(t) \\
& - 4 \left(\frac{nm(n+m)^2(3m^2n + 3mn^2 - 26m^2 + 34mn - 26n^2)}{n^5 + m^5} + 24 \right) H(t)^2H(u) \\
& - 5 \left(\frac{nm(n+m)^2(5m^2n + 5mn^2 - 45m^2 + 59mn - 49n^2)}{n^5 + m^5} + 42 \right) H(s)H(t)H(u) \\
& - 2 \left(\frac{nm(n+m)^2(m^2n + mn^2 - 12m^2 + 16mn - 12n^2)}{n^5 + m^5} + 12 \right) H(s)^2H(u) \\
& - \left(\frac{nm(n+m)^2(5m^2n + 5mn^2 - 45m^2 + 59mn - 49n^2)}{n^5 + m^5} + 42 \right) H(s)H(t)^2 \\
& - \left(\frac{nm(n+m)^2(m^2n + mn^2 - 17m^2 + 23mn - 17n^2)}{n^5 + m^5} + 18 \right) H(s)^2H(t) \\
& + 9 \left(\frac{nm(n+m)^2(3m^2n + 3mn^2 - 26m^2 + 34mn - 26n^2)}{n^5 + m^5} + 24 \right) H(s)H(t)^2H(u) \\
& + 5 \left(\frac{nm(n+m)^2(3m^2n + 3mn^2 - 26m^2 + 34mn - 26n^2)}{n^5 + m^5} + 24 \right) H(s)^2H(t)H(u) \\
& + \left(\frac{nm(n+m)^2(3m^2n + 3mn^2 - 26m^2 + 34mn - 26n^2)}{n^5 + m^5} + 24 \right) H(s)^2H(t)^2 \\
& \left. - 5 \left(\frac{nm(n+m)^2(3m^2n + 3mn^2 - 26m^2 + 34mn - 26n^2)}{n^5 + m^5} + 24 \right) H(s)^2H(t)^2H(u) \right) ds \, dt \, du.
\end{aligned}$$

□

2 Additional Figures

In this section, we present some more figures to complement the results in the main text.

2.1 Approximate distribution of $T_{n,m}$ under H_0

In Section 3.2 of the main manuscript, we show that the distribution of the test statistics under the null hypothesis can be well approximated by the generalised Pareto distribution (GPD). To indicate the good approximation, we present in this section the histograms and quantile-quantile plots (QQ-plots) of the test statistics (under H_0), when the data follow the Normal(0,1), Skewed-Normal(2,2,2), Gamma(5,5), and $N(0,1)^{1-\pi} \cdot N(5,2)^\pi$ (with $\pi \sim \text{Bernoulli}(7/8)$) distributions. They are presented in Figures S1 to S4. All of the figures indicate that the approximation is very good, especially for the test statistics in the rejection regions.

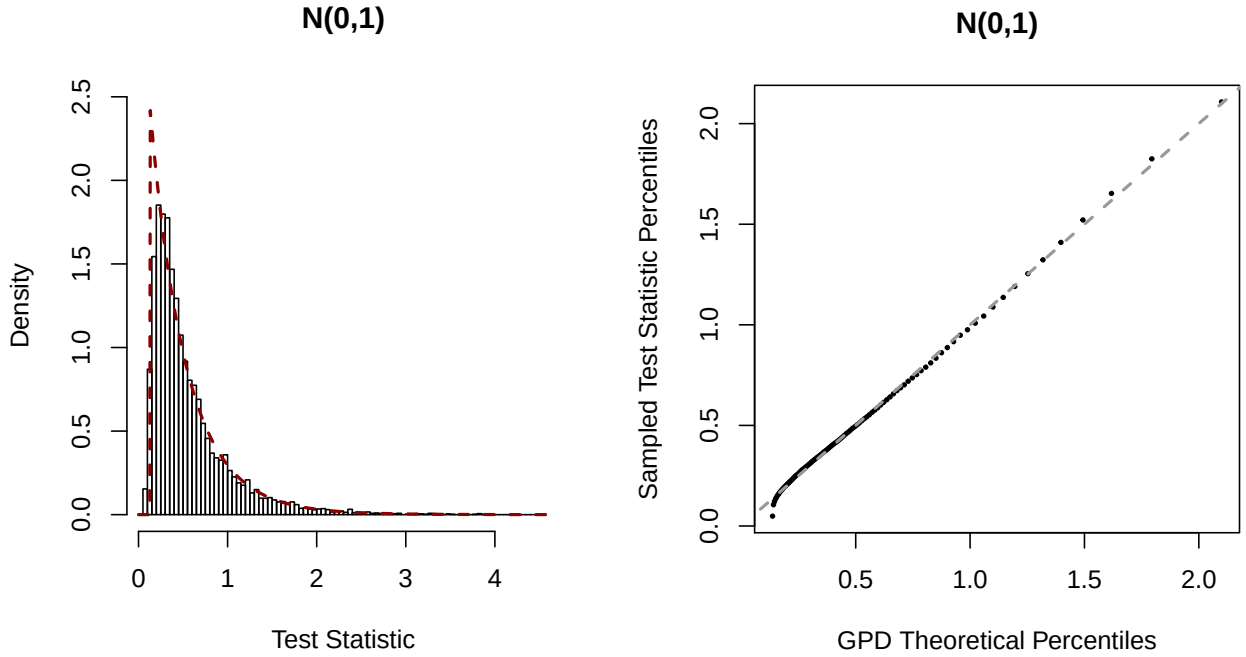


Figure S1: Histogram (left panel) and quantile-quantile plot (QQ-plot, right panel) of the test statistics $T_{n,m}$ under H_0 , when the data are simulated under the Normal(0,1) distribution. The histogram is overlaid with the theoretical generalised Pareto distribution (dashed red line), and the QQ-plot compares the quantiles of the test statistics to the quantiles of the GPD.

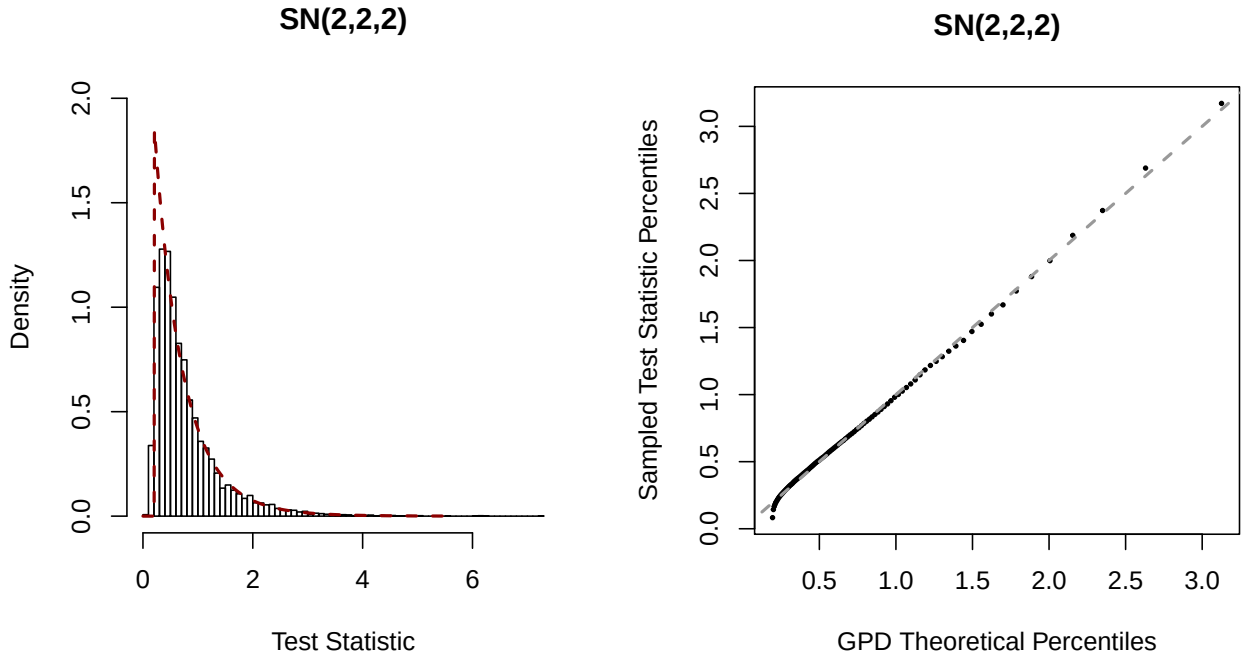


Figure S2: *Histogram (left panel) and quantile-quantile plot (QQ-plot, right panel) of the test statistics $T_{n,m}$ under H_0 , when the data are simulated under the Skewed-Normal(2,2,2) distribution. The histogram is overlaid with the theoretical generalised Pareto distribution (dashed red line), and the QQ-plot compares the quantiles of the test statistics to the quantiles of the GPD.*

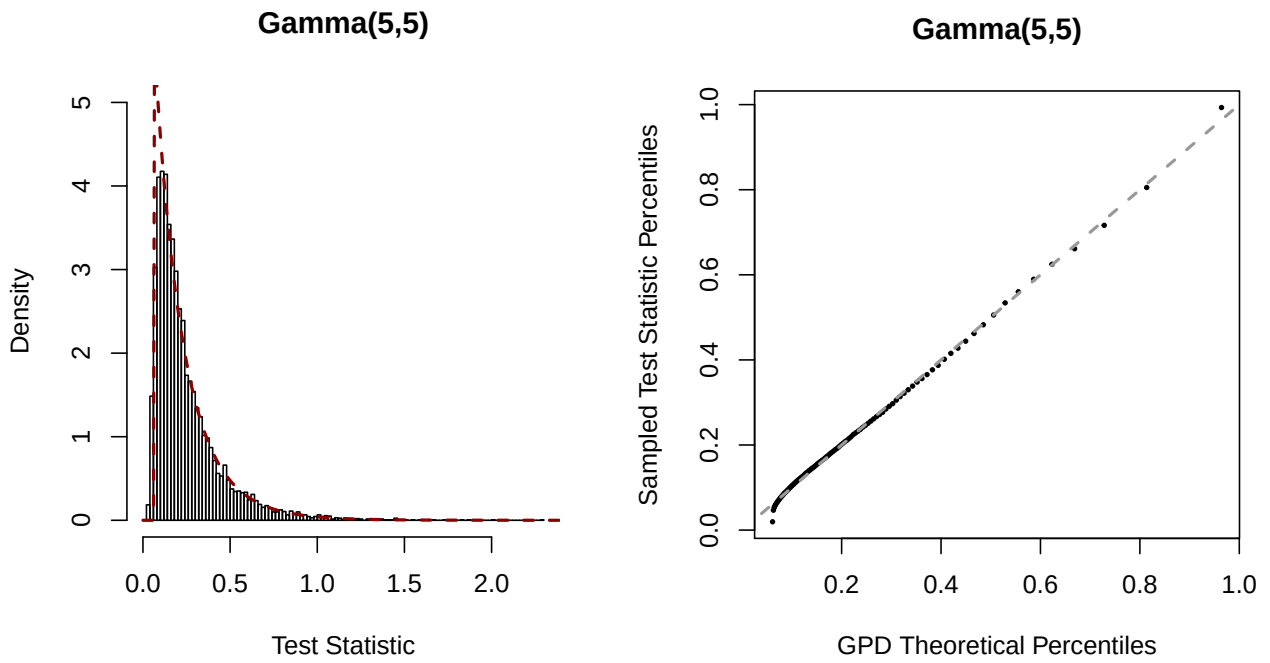


Figure S3: *Histogram (left panel) and quantile-quantile plot (QQ-plot, right panel) of the test statistics $T_{n,m}$ under H_0 , when the data are simulated under the Gamma(5,5) distribution. The histogram is overlaid with the theoretical generalised Pareto distribution (dashed red line), and the QQ-plot compares the quantiles of the test statistics to the quantiles of the GPD.*

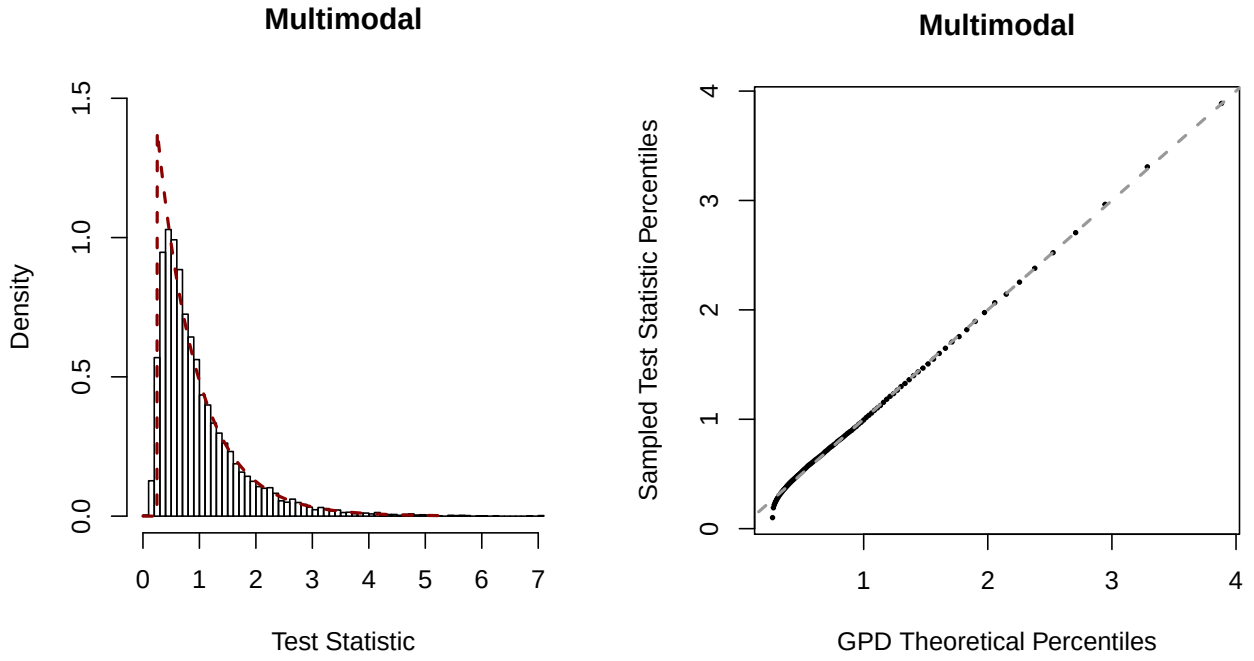


Figure S4: Histogram (left panel) and quantile-quantile plot (QQ-plot, right panel) of the test statistics $T_{n,m}$ under H_0 , when the data are simulated under the $N(0, 1)^{1-\pi} \cdot N(5, 2)^\pi$ (with $\pi \sim \text{Bernoulli}(7/8)$) distribution. The histogram is overlaid with the theoretical generalised Pareto distribution (dashed red line), and the QQ-plot compares the quantiles of the test statistics to the quantiles of the GPD.

3 Type-I error for 0.01 and 0.001 significance levels

In Section 4.1.1 of the main manuscript, we present the results of type-I error control of the test statistics at the 0.05 significance level, where the generalised Pareto distribution (GPD) is the reference distribution. In this section, we present the type-I error control at the 0.01 and 0.001 significance levels in Figure S5 where we vary the mean. The figure indicates that the type-I error control remains appropriate for the corresponding significance level. This is not surprising, since the GPD has a good approximation to the test-statistics' null-distribution as seen in Figures S1 to S4.

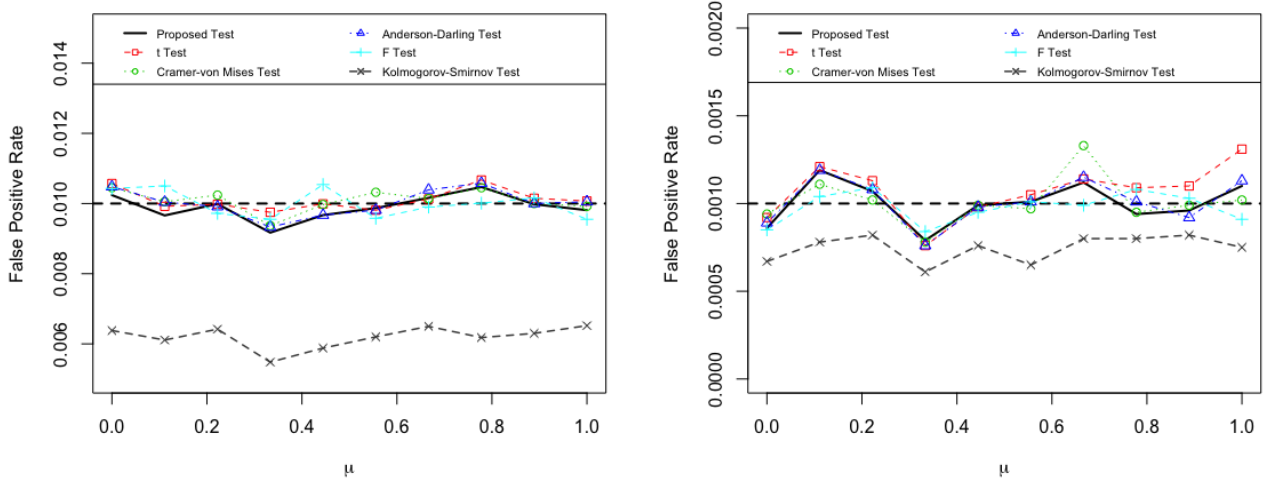


Figure S5: False positive rates for the Cramer test, (two-sample) t -test, Cramer-von Mises test, Anderson Darling test, F -test, and Kolmogorov-Smirnov test at varying μ (from skewed normal $SN(\mu, \sigma^2, \alpha)$) for 0.01 (left panel) and 0.001 (right panel) significance levels.