**Online Supplement for "Condition-based Maintenance for Multi-component systems: Modeling, Structural Properties, and Algorithms" by Zhicheng Zhu and Yisha Xiang**

### A.1. Proof of Proposition 1

***Proof.*** (1) We first consider the case where there is no failed component in the first stage.

We need to compare the total costs among three cases for partition $(N_0, N_1)$: (a) $N_0 = \emptyset$, (b) $N_0 \neq \emptyset$ and $N_1 \neq \emptyset$ and (c) $N_0 = \mathcal{N}$. Denote $C_1$, $C_2$ and $C_3$ by the total costs for the three cases respectively, we show that $C_1$ is minimum.

Denote the total cost for component $i \in \mathcal{N}$ *without* considering economic dependence by

$$\begin{cases} TC_i^1 = c_{i,\text{pm}} + c_\text{s} + Q_i(1,m)(c_{i,\text{cm}} + c_\text{s}), & \tilde{x}_{i,1} = 1, \\ TC_i^0 = Q_i(g_{i,1},m)(c_{i,\text{cm}} + c_\text{s}), & \tilde{x}_{i,1} = 0, \end{cases}$$

Because $\tilde{x}^*_{i,1} = 1$, we have $TC_i^1 < TC_i^0$, $\forall i \in \mathcal{N}$.

Thus, we have

$$\begin{aligned} C_1 &= \sum_{i\in\mathcal{N}} TC_i^1 - (n-1)c_\text{s} - c_\text{s}\sum_{i\in\mathcal{N}} Q_i(1,m) + c_\text{s}(1 - \prod_{i\in\mathcal{N}}(1 - Q_i(1,m)), \\ C_2 &= \sum_{i\in N_0} TC_i^0 + \sum_{i\in N_1} TC_i^1 - (|N_1|-1)c_\text{s} - c_\text{s}(\sum_{i\in N_0} Q_i(g_{i,1},m) + \sum_{i\in N_1} Q_i(1,m)) \\ &\quad + c_\text{s}(1 - \prod_{i\in N_0}(1 - Q_i(g_{i,1},m))\prod_{i\in N_1}(1 - Q_i(1,m))) \text{ and} \\ C_3 &= \sum_{i\in\mathcal{N}} TC_i^0 - c_\text{s}\sum_{i\in\mathcal{N}} Q_i(g_{i,1},m) + c_\text{s}(1 - \prod_{i\in\mathcal{N}}(1 - Q_i(g_{i,1},m)). \end{aligned}$$

(1a) Prove $C_1 < C_2$.

Because

$$\begin{cases} TC_i^0 > TC_i^1 \\ \left(|N_1|-1\right)c_{\mathrm{s}} + c_{\mathrm{s}}\left(\sum_{i\in N_0} Q_i(g_{i,1},m) + \sum_{i\in N_1} Q_i(1,m)\right) < (n-1)c_{\mathrm{s}} + c_{\mathrm{s}}\sum_{i\in\mathcal{N}} Q_i(1,m) \\ c_{\mathrm{s}}(1-\prod_{i\in N_0}(1-Q_i(g_{i,1},m))\prod_{i\in N_1}(1-Q_i(1,m))) > c_{\mathrm{s}}(1-\prod_{i\in\mathcal{N}}(1-Q_i(1,m)) \end{cases}$$

we have $C_1 < C_2$.

(1b) Prove $C_1 < C_3$

It is easy to show that function $f(v_1, v_2, ..., v_n) = \sum_{i\in\mathcal{N}} v_i + \prod_{i\in\mathcal{N}}(1-v_i)$ has $\frac{\partial f}{\partial v_i} \geq 0$ for all $0 \leq v_i \leq 1$, $i \in \mathcal{N}$. Therefore, we have

$$\max(C_1) = C_1|_{Q_i(1,m)=0,\forall i\in\mathcal{N}} = \sum_{i\in\mathcal{N}} TC_i^1 - (n-1)c_{\mathrm{s}}$$

and

$$\min(C_3) = C_3|_{Q_i(g_{i,1},m)=1,\forall i\in\mathcal{N}} = \sum_{i\in\mathcal{N}} TC_i^0 - (n-1)c_{\mathrm{s}}.$$

Because $TC_i^0 > TC_i^1$ for all $i \in \mathcal{N}$, we have $C_1 \leq \max(C_1) < \min(C_3) \leq C_3$.

Therefore, $C_1$ is minimum.

(2) Consider the case where there exists at least one component failed at the first stage.

Let set $N \subseteq \mathcal{N}$ collect all failed components and $N \neq \emptyset$. Following proof (1), we only need to compare case (a) and feasible case (b) because case (c) is not feasible.

The cost of case (a) and feasible case (b) are denoted by $C_1'$ and $C_2'$ respectively, where

$$C_1' = C_1 + \sum_{i\in N}(c_{i,\mathrm{cm}} - c_{i,\mathrm{pm}})$$

and

$$C_2' = C_2 + \sum_{i\in N}(c_{i,\mathrm{cm}} - c_{i,\mathrm{pm}}).$$

From $C_1 < C_2$ in proof (1a), we have $C_1' < C_2'$.

□

### A.2. Proof of Proposition 2

***Proof.*** Denote the total cost for component $i \in \mathcal{N}$ *without* considering economic dependence by

$$\begin{cases} TC_i^1 = c_{i,\mathrm{pm}} + c_\mathrm{s} + Q_i(1,m)(c_{i,\mathrm{cm}} + c_\mathrm{s}), & \tilde{x}_{i,1} = 1, \\ TC_i^0 = Q_i(g_{i,1},m)(c_{i,\mathrm{cm}} + c_\mathrm{s}). & \tilde{x}_{i,1} = 0, \end{cases}$$

and let $Q_i(1,m) = Q_i(1)$ and $Q_i(g_{i,1},m) = Q_i(g)$ $\forall i \in \mathcal{N}$, then we have

$$\begin{aligned} C = &\sum_{i\in N_0} TC_i^0 + \sum_{i\in N_1} TC_i^1 - (\max(|N_1|-1,0))c_\mathrm{s} - c_\mathrm{s}(\sum_{i\in N_0} Q_i(g) + \sum_{i\in N_1} Q_i(1)) \\ &+ c_\mathrm{s}(1 - \prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1))) \end{aligned}$$

$$\begin{aligned} C' = &\sum_{i\in N_0'} TC_i^0 + \sum_{i\in N_1'} TC_i^1 - (\max(|N_1'|-1,0))c_\mathrm{s} \\ &- c_\mathrm{s}(\sum_{i\in N_0'} Q_i(g) + \sum_{i\in N_1'} Q_i(1)) + c_\mathrm{s}(1 - \prod_{i\in N_0'}(1-Q_i(g))\prod_{i\in N_1'}(1-Q_i(1))) \end{aligned}$$

If $N_1 = \emptyset$, we have

$$\begin{aligned} C' - C = &\sum_{k\in N}(TC_k^1 - TC_k^0) + c_\mathrm{s}\sum_{k\in N}(Q_k(g) - Q_k(1)) \\ &+ c_\mathrm{s}(\prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1)) - \prod_{i\in N_0'}(1-Q_i(g))\prod_{i\in N_1'}(1-Q_i(1))) \\ = &\sum_{k\in N}\underbrace{(c_{k,\mathrm{pm}} - (Q_k(g) - Q_k(1))\,c_{k,\mathrm{cm}})}_{\rho_k c_\mathrm{s}} + c_\mathrm{s} \\ &- c_\mathrm{s}\underbrace{\prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1))}_{p(N_0,N_1)}\underbrace{\left(\prod_{k\in N}\frac{1-Q_k(1)}{1-Q_k(g)} - 1\right)}_{r_N} \\ = &\sum_{k\in N}\rho_k c_\mathrm{s} + c_\mathrm{s} - c_\mathrm{s} p(N_0,N_1) r_N \end{aligned}$$

Therefore, from $C' < C$, we have

$$\frac{\sum_{k\in N}\rho_k c_\mathrm{s} + c_\mathrm{s}}{c_\mathrm{s} r_N p(N_0,N_1)} = \frac{1 + \sum_{k\in N}\rho_k}{r_N p(N_0,N_1)} = \Delta_\mathrm{r}(N_0,N_1,N) < 1.$$

From $\Delta_{\mathrm{r}}(N_0, N_1, N) < 1$, we have $C' < C$.

Similarly, if $N_1 \neq \emptyset$,

$$\begin{aligned}
C' - C =& \sum_{k \in N} (TC_k^1 - TC_k^0) - |N| c_{\mathrm{s}} + c_{\mathrm{s}} \sum_{k \in N} (Q_k(g) - Q_k(1)) \\
&+ c_{\mathrm{s}} ( \prod_{i \in N_0} (1 - Q_i(g)) \prod_{i \in N_1} (1 - Q_i(1)) - \prod_{i \in N_0'} (1 - Q_i(g)) \prod_{i \in N_1'} (1 - Q_i(1))) \\
=& \sum_{k \in N} \underbrace{(c_{k,\mathrm{pm}} - (Q_k(g) - Q_k(1))\, c_{k,\mathrm{cm}})}_{\rho_k c_{\mathrm{s}}} \\
&- c_{\mathrm{s}} \underbrace{\prod_{i \in N_0} (1 - Q_i(g)) \prod_{i \in N_1} (1 - Q_i(1))}_{p(N_0, N_1)} \underbrace{\left( \prod_{k \in N} \frac{1 - Q_k(1)}{1 - Q_k(g)} - 1 \right)}_{r_N} \\
=& \sum_{k \in N} \rho_k c_{\mathrm{s}} - c_{\mathrm{s}} p(N_0, N_1) r_N
\end{aligned}$$

Therefore, from $C' < C$, we have

$$\frac{\sum_{k \in N} \rho_k c_{\mathrm{s}}}{c_{\mathrm{s}} r_N p(N_0, N_1)} = \frac{\sum_{k \in N} \rho_k}{r_N p(N_0, N_1)} = \Delta_{\mathrm{r}}(N_0, N_1, N) < 1.$$

From $\Delta_{\mathrm{r}}(N_0, N_1, N) < 1$, we have $C' < C$. □

### A.3. Proof of Proposition 3

***Proof.*** Denote the total cost for component $i \in \mathcal{N}$ *without* considering economic dependence by

$$\begin{cases} TC_i^1 = c_{i,\mathrm{pm}} + c_{\mathrm{s}} + Q_i(1, m)(c_{i,\mathrm{cm}} + c_{\mathrm{s}}), & \tilde{x}_{i,1} = 1, \\ TC_i^0 = Q_i(g_{i,1}, m)(c_{i,\mathrm{cm}} + c_{\mathrm{s}}). & \tilde{x}_{i,1} = 0, \end{cases}$$

and let $Q_i(1, m) = Q_i(1)$ and $Q_i(g_{i,1}, m) = Q_i(g)$ $\forall i \in \mathcal{N}$, then we have

$$\begin{aligned}
C =& \sum_{i \in N_0} TC_i^0 + \sum_{i \in N_1} TC_i^1 - (\max(|N_1| - 1, 0)) c_{\mathrm{s}} - c_{\mathrm{s}} ( \sum_{i \in N_0} Q_i(g) + \sum_{i \in N_1} Q_i(1)) \\
&+ c_{\mathrm{s}} (1 - \prod_{i \in N_0} (1 - Q_i(g)) \prod_{i \in N_1} (1 - Q_i(1)))
\end{aligned}$$

$$
\begin{aligned}
C' = & \sum_{i\in N_0'} TC_i^0 + \sum_{i\in N_1'} TC_i^1 - (\max(|N_1'|-1,0))c_\mathrm{s} - c_\mathrm{s}(\sum_{i\in N_0'} Q_i(g) + \sum_{i\in N_1'} Q_i(1)) \\
& + c_\mathrm{s}(1 - \prod_{i\in N_0'}(1-Q_i(g))\prod_{i\in N_1'}(1-Q_i(1)))
\end{aligned}
$$

If $N_1' = \emptyset$, we have

$$
\begin{aligned}
C - C' = & \sum_{k\in N}(TC_k^1 - TC_k^0) + c_\mathrm{s}\sum_{k\in N}(Q_k(g) - Q_k(1)) \\
& + c_\mathrm{s}(\prod_{i\in N_0'}(1-Q_i(g))\prod_{i\in N_1'}(1-Q_i(1)) - \prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1))) \\
= & \sum_{k\in N}\underbrace{(c_{k,\mathrm{pm}} - (Q_k(g)-Q_k(1))\,c_{k,\mathrm{cm}})}_{\rho_k c_\mathrm{s}} + c_\mathrm{s} \\
& - c_\mathrm{s}\underbrace{\prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1))}_{p(N_0,N_1)}\underbrace{\left(1-\prod_{k\in N}\frac{1-Q_k(g)}{1-Q_k(1)}\right)}_{s_N} \\
= & \sum_{k\in N}\rho_k c_\mathrm{s} + c_\mathrm{s} - c_\mathrm{s} p(N_0,N_1)s_N
\end{aligned}
$$

From $C > C'$, we have $\sum_{k\in N}\rho_k c_\mathrm{s} + c_\mathrm{s} - c_\mathrm{s}p(N_0,N_1)s_N > 0$. Therefore,

$$
\frac{\sum_{k\in N}\rho_k c_\mathrm{s} + c_\mathrm{s}}{c_\mathrm{s}s_N p(N_0,N_1)} = \frac{1+\sum_{k\in N}\rho_k}{s_N p(N_0,N_1)} = \Delta_\mathrm{s}(N_0,N_1,N) > 1,
$$

From $\Delta_\mathrm{s}(N_0,N_1,N) > 1$, we have $C > C'$.

Similarly, if $N_1' \neq \emptyset$,

$$
\begin{aligned}
C - C' = & \sum_{k\in N}(TC_k^1 - TC_k^0) - |N|c_\mathrm{s} + c_\mathrm{s}\sum_{k\in N}(Q_k(g) - Q_k(1)) \\
& + c_\mathrm{s}(\prod_{i\in N_0'}(1-Q_i(g))\prod_{i\in N_1'}(1-Q_i(1)) - \prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1))) \\
= & \sum_{k\in N}\underbrace{(c_{k,\mathrm{pm}} - (Q_k(g)-Q_k(1))\,c_{k,\mathrm{cm}})}_{\rho_k c_\mathrm{s}} \\
& - c_\mathrm{s}\underbrace{\prod_{i\in N_0}(1-Q_i(g))\prod_{i\in N_1}(1-Q_i(1))}_{p(N_0,N_1)}\underbrace{\left(1-\prod_{k\in N}\frac{1-Q_k(g)}{1-Q_k(1)}\right)}_{s_N} \\
= & \sum_{k\in N}\rho_k c_\mathrm{s} - c_\mathrm{s} p(N_0,N_1)s_N
\end{aligned}
$$

From $C > C'$, we have

$$\frac{\sum_{k\in N}\rho_k c_{\mathrm{s}}}{c_{\mathrm{s}} s_N p(N_0,N_1)} = \frac{\sum_{k\in N}\rho_k}{s_N p(N_0,N_1)} = \Delta_{\mathrm{s}}(N_0,N_1,N) > 1,$$

From $\Delta_{\mathrm{s}}(N_0,N_1,N) > 1$, we have $C > C'$ □

### A.4. Proof of Proposition 4

***Proof.*** We prove this proposition by showing that the cost of partition $(N_0^*, N_1^*)$ is no worse than that of any other feasible partitions.

For any other feasible partition $(N_0', N_1')$ and the partition $(N_0^*, N_1^*)$ that is obtained by Algorithm 1, we always rewrite $(N_0', N_1') = (N_0 \cup N_b, N_1 \cup N_a)$ and $(N_0^*, N_1^*) = (N_0 \cup N_a, N_1 \cup N_b)$ respectively, where set $N_0 = N_0' \cap N_0^*$, $N_1 = N_1' \cap N_1^*$, $N_b = N_0' \backslash N_0 = N_1^* \backslash N_1$ and $N_a = N_1' \backslash N_1 = N_0^* \backslash N_0$. We now show that the cost of partition $(N_0^*, N_1^*)$ is no worse than that of $(N_0', N_1')$ by the following three parts: (1) When $N_b \neq \emptyset$, we have cost relationship $(N_0^*, N_1^*) = (N_0 \cup N_a, N_1 \cup N_b) < (N_0 \cup N_a \cup N_b, N_1)$, (2) when $N_b \neq \emptyset$, we have cost relationship $(N_0 \cup N_a \cup N_b, N_1) < (N_0 \cup N_b, N_1 \cup N_a) = (N_0', N_1')$, and (3) we have cost $(N_0', N_1') = (N_0^*, N_1^*)$ if and only if $N_b = \emptyset$ and $\Delta_{\mathrm{r}}(N_0 \cup N_a, N_1, N_a) = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_a, N_a) = 1$.

(1) When $N_b \neq \emptyset$, we have cost relationship $(N_0^*, N_1^*) = (N_0 \cup N_a, N_1 \cup N_b) < (N_0 \cup N_a \cup N_b, N_1)$.

This is equivalent to show that given current partition $(N_0 \cup N_a \cup N_b, N_1)$, moving $N_b$ from the do-nothing set to the maintenance set can reduce cost. We next show that if we keep moving the component that arrives first in $N_b$ in Algorithm 1 to the maintenance set, the cost keeps reducing until $N_b = \emptyset$, which implies moving the whole set $N_b$ to the maintenance set reduces cost.

Denote the costs of $(N_0 \cup N_a, N_1 \cup N_b)$ and $(N_0 \cup N_a \cup N_b, N_1)$ by $C$ and $C_0$ respectively, and initialize $C' = C_0$. We prove $C < C_0$ by the following steps:

**Step 1**: If all components in $N_b$ are moved into $N_1^*$ after set $N_1$ does in Algorithm 1, then $C < C_0$ because the cost reduces if we repeat how Algorithm 1 moves $N_b$ to $N_1^*$.

**Step 2**: In this step, there exists at least one component $i \in N_b$ that joins $N_1^*$ no later than some component in $N_1$. Suppose component $k \in N_b$ is the earliest one in $N_b$ that joins $N_1^*$ and suppose $k$ joins $N_1^*$ along with set $S^j$, i.e., $k \in S^j$, where $|S^j| = j$ and $S^j \subseteq N_1^*$. Therefore, when $S^j \subseteq N_1^*$ joins $N_1^*$, the current partition is $(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S)$, where set $S^j \backslash \{k\} \subseteq S$, and hence from Proposition 2, we have

$$\Delta_{\mathrm{r}}(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S, S^j) < 1. \tag{1}$$

**Step 3**: If $j = 1$, then $S^j = \{k\}$. Denote the costs for partition $(N_0 \cup N_a \cup N_b \backslash \{k\}, N_1 \cup \{k\})$ by $C_1$. From Inequation (1), we have $\Delta_{\mathrm{r}}(N_0 \cup N_a \cup N_b, N_1, \{k\}) < \Delta_{\mathrm{r}}(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S, \{k\}) < 1$ and therefore $C_1 < C'$. We then update $N_b = N_b \backslash \{k\}$ and $C' = C_1$ and go to Step 1.

**Step 4**: In this step, we have $j > 1$. From Algorithm 1, we know that any subset $N \subset S^j$ cannot join $N_1^*$ given current partition $(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S)$. From Proposition 2, we have

$$\Delta_{\mathrm{r}}(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S, N) > 1. \tag{2}$$

Let $N + \{k\} = S^j$. We have

$$r_N p(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S) + \rho_k < \sum_{i \in N} \rho_i + \rho_k < r_{S^j} p(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S),$$

where the first inequality is from Inequation (2) and the second inequality is from Inequation (1). Therefore, we have $\rho_k < (r_{S^j} - r_N) p(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S)$ and hence

$$\begin{aligned}
\Delta_{\mathrm{r}}(N_0 \cup N_a \cup N_b, N_1, \{k\}) &= \frac{\rho_k}{r_{\{k\}} p(N_0 \cup N_a \cup N_b, N_1)} \\
&< \frac{(r_{S^j} - r_N) p(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S)}{r_{\{k\}} p(N_0 \cup N_a \cup N_b, N_1)} \\
&= \frac{\prod_{i \in N} \frac{1 - Q_i(1,m)}{1 - Q_i(g_{i,1},m)} p(N_0 \cup N_a \cup N_b \cup S, N_1 \backslash S)}{p(N_0 \cup N_a \cup N_b, N_1)} \\
&= \frac{p(N_0 \cup N_a \cup N_b \cup S \backslash N, (N_1 \cup N) \backslash S)}{p(N_0 \cup N_a \cup N_b, N_1)} \leq 1,
\end{aligned}$$

where the last inequality is from $N = S^j \backslash \{k\} \subseteq S$. From Proposition 2, by denoting the cost of partition $(N_0 \cup N_a \cup N_b \backslash \{k\}, N_1 \cup \{k\})$ by $C_1$, we have $C_1 < C'$ since $\Delta_{\mathrm{r}}(N_0 \cup N_a \cup N_b, N_1, \{k\}) < 1$. We then update $N_b = N_b \backslash \{k\}$ and $C' = C_1$ and go to Step 1.

Therefore, we can always lower the cost $C'$ by moving one component from $N_b$ to the maintenance set. When $N_b = \emptyset$, we have $C' = C < C_0$.

(2) When $N_b \neq \emptyset$, we have cost relationship $(N_0 \cup N_a \cup N_b, N_1) < (N_0 \cup N_b, N_1 \cup N_a) = (N_0', N_1')$.

This is equivalent to show that moving set $N_a$ from the maintenance set to the do-nothing set can lower cost. From Proposition 3, we need to prove $\Delta_{\mathrm{s}}(N_0 \cup N_b, N_1 \cup N_a, N_a) > 1$.

By using the same method as proof (1), we can also prove the cost relationship $(N_0 \cup N_a, N_1 \cup N_b) < (N_0, N_1 \cup N_a \cup N_b)$ when $N_b \neq \emptyset$. From Proposition 3, we have $\Delta_{\mathrm{s}}(N_0, N_1 \cup N_a \cup N_b, N_a) > 1$. Therefore,

$$\Delta_{\mathrm{s}}(N_0 \cup N_b, N_1 \cup N_a, N_a) > \Delta_{\mathrm{s}}(N_0, N_1 \cup N_a \cup N_b, N_a) > 1$$

(3) We have cost $(N_0', N_1') = (N_0^*, N_1^*)$ if and only if $N_b = \emptyset$ and $\Delta_{\mathrm{r}}(N_0 \cup N_a, N_1, N_a) = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_a, N_a) = 1$.

When $(N_0^*, N_1^*) = (N_0', N_1')$, we have $(N_0 \cup N_a, N_1 \cup N_b) = (N_0 \cup N_a \cup N_b, N_1) = (N_0 \cup N_b, N_1 \cup N_a)$.

The first equality $(N_0 \cup N_a, N_1 \cup N_b) = (N_0 \cup N_a \cup N_b, N_1)$ holds if and only if $N_b = \emptyset$. Otherwise, following the steps of proof (1), we can always have $(N_0 \cup N_a, N_1 \cup N_b) < (N_0 \cup N_a \cup N_b, N_1)$.

Given $N_b = \emptyset$, the second equality is equivalent to $(N_0 \cup N_a, N_1) = (N_0, N_1 \cup N_a)$, which happens if and only if $\Delta_{\mathrm{r}}(N_0 \cup N_a, N_1, N_a) = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_a, N_a) = 1$ based on Corollary 1.

□

**B.1. Proof of Corollary 1:**

***Proof.*** We first show $p(N_0 \cup N_{\mathrm{u}}, N_1) r_N \leq p(N_0, N_1 \cup N_{\mathrm{u}}) s_N$.

$$\begin{aligned}
&p(N_0 \cup N_{\mathrm{u}}, N_1) r_N \\
=& \prod_{i \in N_0 \cup N_{\mathrm{u}}} (1 - Q_i(g_{i,1}, m)) \prod_{i \in N_1} (1 - Q_i(1, m)) \left( \frac{\prod_{i \in N}(1 - Q_i(1, m)) - \prod_{i \in N}(1 - Q_i(g_{i,1}, m))}{\prod_{i \in N}(1 - Q_i(g_{i,1}, m))} \right) \\
=& \prod_{i \in N_0 \cup N_{\mathrm{u}} - N} (1 - Q_i(g_{i,1}, m)) \prod_{i \in N_1 \cup N} (1 - Q_i(1, m)) \left( \frac{\prod_{i \in N}(1 - Q_i(1, m)) - \prod_{i \in N}(1 - Q_i(g_{i,1}, m))}{\prod_{i \in N}(1 - Q_i(1, m))} \right) \\
=& p(N_0 \cup N_{\mathrm{u}} \backslash N, N_1 \cup N) s_N \leq p(N_0, N_1 \cup N_{\mathrm{u}}) s_N,
\end{aligned}$$

where equality holds when $N = N_{\mathrm{u}}$.

(1) When $N_1 \neq \emptyset$, we have

$$\Delta_{\mathrm{r}}(N_0 \cup N_{\mathrm{u}}, N_1, N) = \frac{\sum_{k \in N} \rho_k}{r_N p(N_0 \cup N_{\mathrm{u}}, N_1)} \geq \frac{\sum_{k \in N} \rho_k}{s_N p(N_0, N_1 \cup N_{\mathrm{u}})} = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_{\mathrm{u}}, N),$$

where equality holds when $N = N_{\mathrm{u}}$.

(2) When $N_1 = \emptyset$ and $N \neq N_{\mathrm{u}}$, we have

$$\Delta_{\mathrm{r}}(N_0 \cup N_{\mathrm{u}}, N_1, N) = \frac{1 + \sum_{k \in N} \rho_k}{r_N p(N_0 \cup N_{\mathrm{u}}, N_1)} > \frac{\sum_{k \in N} \rho_k}{s_N p(N_0, N_1 \cup N_{\mathrm{u}})} = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_{\mathrm{u}}, N).$$

(3)When $N_1 = \emptyset$ and $N = N_{\mathrm{u}}$, we have

$$\Delta_{\mathrm{r}}(N_0 \cup N_{\mathrm{u}}, N_1, N) = \frac{1 + \sum_{k \in N} \rho_k}{r_N p(N_0 \cup N_{\mathrm{u}}, N_1)} = \frac{1 + \sum_{k \in N} \rho_k}{s_N p(N_0, N_1 \cup N_{\mathrm{u}})} = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_{\mathrm{u}}, N).$$

Therefore, $\Delta_{\mathrm{r}}(N_0 \cup N_{\mathrm{u}}, N_1, N) > \Delta_{\mathrm{s}}(N_0, N_1 \cup N_{\mathrm{u}}, N)$ when $N \subset N_{\mathrm{u}}$ and $\Delta_{\mathrm{r}}(N_0 \cup N_{\mathrm{u}}, N_1, N) = \Delta_{\mathrm{s}}(N_0, N_1 \cup N_{\mathrm{u}}, N)$ when $N = N_{\mathrm{u}}$. $\square$