APPENDIX A: TEACHING MATERIALS

We include below a sample "Information Sheet", Pre-class Activity and Worksheet on limits. We show here the instructor version of the worksheet, meaning that the comments in blue are for the instructors. The student version is similar but does not contain these comments.
The complete set of worksheets is available at: http://pi.math.cornell.edu/~activelearn.

Limits and One-sided Limits (2.1, 2.2 & 2.4 in Thomas)

Expected Skills.
At the end of these sections, the students will be able to:

- explain in their own words the definition of a limit and one-sided limit,
- give examples that illustrate the different cases where a limit or a one-sided limit fails to exist,
- explain why $\lim_{x \to 0} \sin(1/x)$ does not exist.

Pre-Class Activity
The goal of this pre-class activity is to introduce the students to the concept of a limit and have them realize the following points:

- even though the functions presented in the exercise are not defined everywhere, if we plug in points that are close to the point where they are undefined, we observe that these functions do go to a specific value (in that sense, the function follow a "pattern"),
- even though the function never attains a certain value (e.g. $f(x)$ never equals 3), we can have $f(x)$ get as close as we want to 3.
  In other words, we can decide on a pre-determined level of precision with which we want the function to "attain" the value 3. We can make this level of precision as small as we want (make sure to underline this last point in class!),
- in general, the function can return values that are both below and above the limit (i.e. in general a limit is not an upper or lower bound),
- the interval for which we can assure that the function is going to be in this pre-determined level of precision we want depends on both the function and the level of precision.

Aspects that are treated here and that will need to be investigated in class include:

- for the limit to exist, we need to functions to go to a *single* value no matter how we go to the point (as opposed to cases where the left-hand side and right-hand side limits are different, or where the value of the functions depends on how we approach the point a),
- we can also compute limits for points where the functions is defined (and the actual value of the function does NOT matter).

Worksheet
In the pre-class activity we have the students think about the concept of a limit by working on examples where the limit exists. We also introduce there the idea of pre-determined level of precision.
In this activity, we start by giving two examples where the limit does not exist. The goal is to have the students think about what the definition of a limit should and should not be. We then introduce the definition and ask the students to summarize the main points.
Part 4 focuses on the difference between the limit and the value of the function.
Finally we introduce examples where the limit does not exist and introduce the definition of a one-sided limit.

Limits (Thomas 2.1-2.2) MATH 1110
Due: ADD DATE Pre-Class Activity

Here we will look at the concept of a limit from a general point of view. In class we will investigate more precisely the definition of a limit and what its various subtleties are.

Consider the function: $f(x) = \dfrac{\sqrt{x^2+4}-2}{x^2}$.

1. Determine the domain of $f(x)$ and identify any x-values at which it is *not* defined.

2. If a function is undefined at a point $x = a$, it means that we cannot plug this value into the function. Nevertheless, we *may be able to* compute the function at x-values that are very close to a.

   Compute the value of the function $f(x)$ for some x-values that very close to the point where it is undefined. What do you notice? Do you see a pattern?

A conclusion we can draw from the previous parts is that even though the function $f(x)$ is not defined at $x = 0$, if we plug in x-values that are very close to $x = 0$, the function values are all very close to one specific number. We will investigate this aspect more precisely now.

3. Use Geogebra or Desmos to graph the function $f(x)$. Does the graph correspond to your computations? Is the graph correct (warning: graphing software often has the bad habit of filling in holes)?

4. For the function $f(x)$ there is no x-value a such that $f(a) = \frac{1}{4}$. One important question is the following: even though $f(x)$ never gives the exact value $\frac{1}{4}$, does the function ever get as close as we want to $\frac{1}{4}$? Let's say we want $f(x)$ to be near the value $\frac{1}{4}$ with a level of precision of 0.1. Find several values of x for which $0.15 < f(x) < 0.35$.

5. Can you find an interval around $x = 0$ on which you can assure that $f(x)$ is between 0.15 and 0.35?

6. Let us now do the same with a level of precision of 0.01: we now want $f(x)$ to be between 0.24 and 0.26. Can you find an interval around $x = 0$ for which you can assure that $f(x)$ is going to be between 0.24 and 0.26? Does the interval you found at the previous point also work here?

1. What are the 3 or 4 main elements you retain from the pre-class activity?

   - *even though the functions presented in the exercise are not defined everywhere, if we plug in points that are close to the point where they are undefined, we observe that these functions do go to a **specific value** (in that sense, the function follow a "**pattern**"),*
   - *even though the function never attains a certain value (e.g. $f(x)$ never equals 3), we can have $f(x)$ get **as close as we want** to 3.*
     *In other words, we can decide on a **pre-determined precision** with which we want the function to attain the value 3. We can make this level of precision as small as we want (make sure to underline this last point in class!),*
   - *in general, the function can return values that are **both below and above** the limit value (i.e. in general a limit is not an upper or lower bound),*
   - *the interval for which we can assure that the function is going to be in the range we want **depends on both the function and the level of precision**.*

   Let us keep in mind this idea of having a pre-determined level of precision around *one* value. We will investigate a few more examples and then will get to the definition of a limit.

2. Consider the following functions:

$$f(x) = \frac{x}{|x|}, \qquad g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \text{ for } n \text{ non-zero integers,} \\ 0 & \text{if } x \neq \frac{1}{n}, \text{ for } n \text{ non-zero integers.} \end{cases}$$

   (a) What are the domains of definition of these two functions? Sketch their graphs.

   If you want to be more precise, you can say that f(x) is undefined at 0. Nevertheless, it would be more interesting to see if students come up with this question (and if not to prompt them about it).

(b) Concerning $f(x)$, what value(s) does the function go to if we take points close to $x = 0$?

It depends what side one approaches 0 from!
The function $f(x)$ goes to -1 from the left-hand side and 1 from the right-hand side.

(c) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?
No, BUT we can do that for each side individually.

Key point(s) of this example

For some functions, the behavior of the function depends one the side one approaches from.
We will thus need more than one notion to capture these different behaviors (here one-sided limits).

(d) Let us now look at $g(x)$. What value(s) does the function go to if we take points close to $x = 0$?

It depends on "how" you approach 0!
If one approaches 0 following points that are all different from $1/n$ we always find $g(x) = 0$ whereas if one approaches 0 following $1/n$ we always get $g(x) = 1$.

(e) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?

Explain why it is not possible (this will be a case where the "limit does not exist".

Key point(s) of this example

For some functions one ***cannot*** *find an interval such that the function remains in a pre-determined range.*

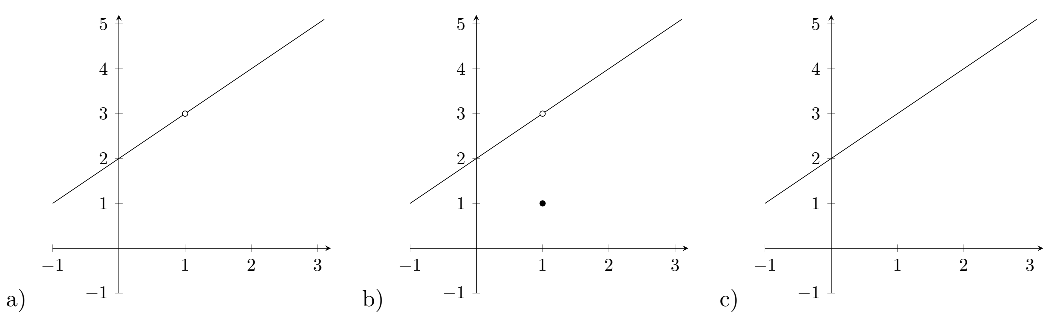
Limits
We can think of the *limit of* $f(x)$ *as* x *approaches* a in the following way: choose *any* pre-determined level of precision. Then the limit $\lim_{x\to a} f(x)$ equals L if we can find an interval around a, such that for any x different from a in this interval, the function $f(x)$ approaches L with the desired pre-determined level of precision. To make it short, we use the notation $\lim_{x\to a} f(x)$ for the limit of $f(x)$ as x approaches a.
Note that we use the non-technical expression "the function $f(x)$ approaches L" in order to avoid confusing students with jargon. Underline that this is really the idea of distance that we mean here (the distance between $f(x)$ and L is smaller than the pre-determined level of precision).

3. What are the key elements of this definition?

   - *the limit is a* ***single number****,*
   - *there are cases where the limit* ***does not exist*** *(we will investigate that soon), have we already encountered such cases?*
   - *since $x \neq a$, the function* ***may or may not be defined*** *at $x = a$ (so far we have mainly looked at examples where the function was not defined at a). If it is defined, the value $f(a)$ of the function at $x = a$* ***does NOT matter*** *for the limit*
   - *we can make the level of prevision* ***as small as we want****. Once we have done that, what we need to do if find an interval around a for which all values of $f(x)$ are inside this level of precision (except at a itself)*

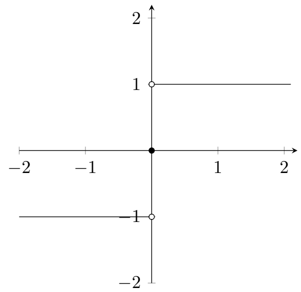
   Using the graphs of the previous functions could be helpful here!

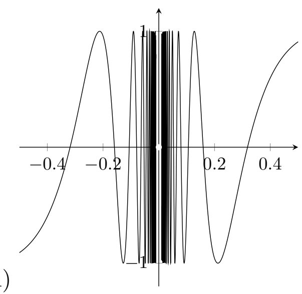
4. Let us now look at what this means graphically. For each of the following example, determine $\lim_{x\to 1} f(x)$ as well as $f(1)$.

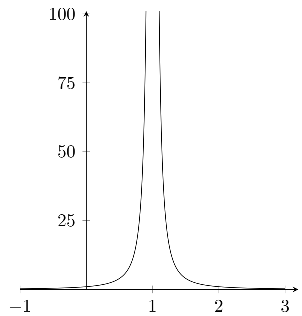


Conclusion: whether or not the function is defined has no influence on the limit.
Moreover, $f(a)$ and $\lim_{x\to a} f(x)$ may be different.

5. Let us now consider the following functions. For i) and ii), determine if the limit $\lim_{x \to 0} f(x)$ exists and if so, what it is. Determine also $f(0)$. For iii), same questions but for $\lim_{x \to 1} f(x)$ and $f(1)$.
   Note that for ii) the function oscillates more and more as x goes to 0.

   i)
   

   ii)
   

   iii)
   

   Student may (and should!) have questions about the graphs ii) and iii), this is a good thing, encourage them!!
   For ii), the function is undefined at $x = 0$ (the function being $\sin(1/x)$). For iii), the function keeps increasing as x goes to 1 (the function being $\frac{1}{(x-1)^2}$).

   i) $f(0) = 0$ and $\lim_{x \to 0} f(x)$ does not exist because the one-sided limits are different,
   ii) $f(0)$ is undefined and $\lim_{x \to 0} f(x)$ does not exist because of the oscillations,
   iii) $f(1)$ is undefined and $\lim_{x \to 1} f(x)$ does not exist because it is unbounded (will be studied again later).

   Overall, these are illustrations of the ways in which a limit can fail to exist.

The first example above motivates the following definition of **one-sided limits**.
For *any* pre-determined level of precision we choose,

- the *limit of $f(x)$ as x approaches a from the left*, written $\lim_{x \to a^-} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (b, a) with $b < a$, in other words with x strictly *smaller* than a.
- the *limit of $f(x)$ as x approaches a from the right*, written $\lim_{x \to a^+} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (a, b) with $a < b$, in other words with x strictly *greater* than a.

APPENDIX B: EXERPT FROM THE INSTRUCTOR MANUAL

Below is the "Teaching Style" section of the Instructor Manual for MATH 1110. This section was added in summer 2018 after the first iteration of the course using the worksheets.

Teaching style

Math 1110 has been one focus of the Math Department's *Active Learning Initiative*. We have produced some materials to help you structure your lessons. We are not fully "flipping" the classroom (which might mean never, ever lecturing). Rather, we have materials that should allow you to break up class time so that students are spending a significant portion of the class period **doing mathematics** and **getting your feedback**. That will be mixed in with periods of you writing careful statements (definitions, examples, theorems) on the board and giving some mini-lectures.

Get to know your students. Learn your students' names! The class lists with pictures on Faculty Center are a help. Be approachable, understanding and empathetic. Be friendly, but professional.

Class Structure: In order to create an active learning environment, there will be pre-class exercises to complete at home. Classe time will consist of working through selected questions on worksheets, with instructors lecturing for short periods in between. The pre-class and worksheets have been written largely by Steve Bennoun. These will be made available for all instructors in the shared Box Folder. Please feel free to adapt them for your section's needs.

The first class of the semester

The first class sets the tone for the semester. Calculus I plays a particularly crucial role (gatekeeper?) role in undergraduate education and has been identified as one of the "pipeline leaks" leading to in the STEM gender gap. On Box under Preclass+Worksheets you can find some first day of class activities.

First 20 minutes

- Introduce yourself (and write your name, the course and section numbers on the board or slides).
- Take attendance somehow – either by calling names or passing a sheet around to sign or assigning something to hand in, see Box for first day questionnaire.
- Distribute the syllabus or a handout with course policies (but you do not need to read it to them!)
- Discuss goals and expectations for the semester, including
  - This is a 4-credit course (which means to expect spending 9–12 hours on it, outside of class!)
  - Reading the text and working on pre-class exercises
  - Active and Cooperative Learning
  - Written and Verbal explanations
  - Homework and quizzes (and dropped grades)
  - Participation
  - Surveys
  - Common Evening Prelims and Final (with DATES!)

- Talk about **transferable skills**, valuable skills that are useful in other contexts can be gained from a calculus course. These include the ability to assimilate abstract or technical ideas, imagination and visualization, composing a careful argument, and clear and accurate writing. Try to model and teach such skills in the classes, homework, exams etc..
- You might want to establish class norms with your students, I give a few to my students then ask them to add their own:
  - There are no stupid questions, each of your questions and answers can help us learn more.
  - It is encouraged to be out of our comfort zone, that is when we learn.
  - What about cell phones and other technology?
- You might try a *gallery walk*, as described in this post on the Art of Mathematics site. The "icebreaker.pdf" files contains some of these questions that I like to ask to my classes.

Next 20 minutes

- Get the students to make groups of 3-4 students. Our classrooms are so full that we don't have a good way of getting students to mix up. Though maybe in the second class, have them move seats, so they get the opportunity to work with other students in the class.
- Start with a brief discussion of the concept of $\boxed{\text{function}}$. Emphasize from the outset that a function is nothing but a rule that takes certain numbers as inputs and for each input assigns exactly one output number: this rule is not necessarily a "math formula." Students often believe functions and formulas are one and the same, so lead off with examples that de-emphasize formulas in favor of tables, graphs, and verbal descriptions.
- Try out the introductory activity from our shared Box folder.

Last 10 minutes

- If you have time, lead a class discussion beginning a review of precalculus material (Chapter 1 in the textbook)
- Remind students of available resources (as listed on Canvas) – they may now recognize the need to attend a Math Support Center Review session, for example!
- Be certain to allow time before the end of class to come to closure and recap any of the problems you assign. Remind the class of the reading, pre-class assignments and homework, including the surveys linked from Blackboard. Re-stress that it is critical to read the book and work on the pre-class exercises before they come to class, as you will not re-teach everything in the reading.
- Ask the students to complete the questionnaire, give them at least 5 minutes for this.
- If you have a 75 minute lecture, you might have students work on a cooperative quiz reviewing precalculus materials. There are some precalculus review questions and solutions (courtesy of Iian Smythe) linked off this page, which you could suggest the students look at if you wish.

At the beginning of the **second class**, please make sure students have succeeded in accessing Blackboard, WeBWorK, and Gradescope.

Teaching Strategies

Here are some additional strategies you can try to foster active engagement with mathematics in your classroom.

- **Think-Pair-Share**

  The teacher asks a question. The students first think by themselves about it and write down their answer, then discuss their answer with a neighbor, and finally engage in class discussion.

- **"Simple" class polling**

  The instructor asks a question and gathers answers from all the students with or without technology.

- **Peer Instruction (with Clickers)**

  Students first answer a multiple-choice question using a student response system. They then try to convince one another that their answer is correct. After some discussion, they answer the same question again.

  Clickers are small key pads which allow students to submit an answer to a multiple-choice question in class, and for the instructor to display a bar chart of the responses via a laptop and projector. They can be a useful tool for provoking class discussions I suggest that if you use them, do so informally—that is, don't keep track of the students responses; to do so would introduce significant logistical challenges. Dick Furnas keeps a supply of clickers and base units. Some students have clickers as they are required for other classes, so you may find you don't need many.

  A good strategy for using the clickers is to get a first round of clicker responses to a question and display the results. Then, if there is between 35% and 75% of correct answers, invite the students to debate the answer with their neighbors. Then take a second round of responses, which hopefully will have moved towards the correct answer. Finally give the correct answer and some explanation. Clickers can be an excellent way of discovering how much your students are understanding. To save time, instructors using clickers may like to collaborate on composing clicker questions. Two or three clicker questions per class are likely to be plenty.

  There are also other variations of clickers such as Plickers which require the instructor to have a smart phone and print some cards, and don't require the students to buy any extra material. An even simpler alternative is to give each student 4 small colored pieces of paper so they can vote in class. This technique can be made more anonymous by making the back of the cards white. It makes interaction and voting very simple, but unfortunately does not store the data.

  There are also cell-phone clicker like applications such as Pingo, Top Hat, Web Clicker, Soap Box, Poll Everywhere and Socrative. A major downside of these is that your are encouraging your students to have their phone out during class.

- **Minute Paper**

  Short in-class activity during which students answer a question anonymously on a piece of paper that is then collected by the instructor. This is a good way to get feedback on where the students are. You might ask, "What is one important idea that we discussed in class today?" or "What idea have you found most confusing so far this week?"

Whilst attempting to provide help and support, make it clear that, in the end, students must take responsibility for their understanding. Mathematics cannot be learnt passively. To quote Paul Halmos:

> *Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?*

Fire these sorts of questions at the students (or rhetorically at yourself) during class. Try to demonstrate thought-processes that lead to solutions. Resist the idea that calculus is a collection of algorithms to be learnt by rote. Many students will have got used to a recipe-style approach to mathematics in high-school. We want to present it more conceptually.

Devoting class time to these 'active learning' techniques helps with knowledge *assimilation*, but cuts the time available for knowledge *transfer*. But our students have reading assignments, online homework, and written homework to fill the gaps.