Semiparametric Tail Index Regression

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Supplementary Material

The supplementary materials include several lemmas and the details of proof for Theorems 1 and Theorem 2 in our article titled with " Semiparametric Tail Index Regression".

S1 Several lemmas

Lemma 1. *Suppose that conditions (C1)–(C5) are satisfied, then for $k = 0, \ldots, 4$ and $n_0 \to \infty$, we have*

$$\sup_{z\in\mathcal{Z}} \left| \frac{1}{n_0 h_n} \sum_{i=1}^{n} K\left(\frac{Z_i - z}{h_n}\right)\left(\frac{Z_i - z}{h_n}\right)^k - f_Z(z)\mu_k \right| = O_p\left\{h_n^2 + \left(\frac{\log n_0}{n_0 h_n}\right)^{1/2}\right\}.$$

Proof. Lemma 1 follows immediately from the results of Mack and Silverman (1982). □

Lemma 2. *If the conditions in Lemma 1 are satisfied, then as $n \to \infty$,*

$$\sup_{1\le i\le n} |\tilde{\eta}(Z_i)| = O_p(h_n^2)$$

where $(\tilde{\eta}(Z_1), \ldots, \tilde{\eta}(Z_n))^\top = (\mathbf{I} - \mathbf{S})(\eta(Z_1), \ldots, \eta(Z_n))^\top$ and $\mathbf{S}$ is the smoothing matrix commonly used in local linear regression that can be referred in Fan and Gijbels (1996).

Proof. Following the definition of $\tilde{\eta}(Z_i)$, we get

$$\tilde{\eta}(Z_i) = \eta(Z_i) - \sum_{i_1=1}^{n} \omega_{i_1}(Z_i)\eta(Z_{i_1})$$

where

$$\omega_{i_1}(Z_i) = K((Z_{i_1} - Z_i)/h_n)\{D_2(Z_i) - (Z_{i_1} - Z_i)\}/\{D_2(Z_i)D_0(Z_i) - D_1^2(Z_i)\}$$

is the local linear weight and $D_s(z) = \sum_{i_1=1}^{n}(Z_{i_1} - z)^s K((Z_{i_1} - z)/h_n)$ with $s = 1, 2$. Integrating Lemma 1 with $\omega_i(z) = (f(Z_i))^{-1}K((Z_i - z)/h_n)/(n_0 h_n) + O_p\{(n_0 h_n)^{-2}\}$ uniformly over $(h_n, 1 - h_n)$ leads to the conclusion in Lemma 2. □

S2 Proofs of main results

Proof of Theorem 1. We write $\boldsymbol{\delta}_0(\boldsymbol{\theta}_0, z) = (\eta_0(z), \eta_0^{(1)}(z))^\top$ for simplicity. Simple calculations using a Taylor expansion show that

$$\begin{aligned}
&\sqrt{n_0 h_n}\left\{\widehat{\boldsymbol{\delta}}_n(\widehat{\boldsymbol{\theta}}_n, z) - \boldsymbol{\delta}_0(\boldsymbol{\theta}_0, z)\right\} \\
=&\sqrt{n_0 h_n}\left\{\widehat{\boldsymbol{\delta}}_n(\widehat{\boldsymbol{\theta}}_n, z) - \widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}_0, z)\right\} + \sqrt{n_0 h_n}\left\{\widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}_0, z) - \boldsymbol{\delta}_0(\boldsymbol{\theta}_0, z)\right\} \\
=&\sqrt{h_n}\widehat{\boldsymbol{\delta}}_n^{(1)}(\widehat{\boldsymbol{\theta}}_n, z)\sqrt{n_0}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) + \sqrt{n_0 h_n}\widehat{\boldsymbol{\delta}}_n^{(2)}(\widehat{\boldsymbol{\theta}}_n, z)O_p(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)^2 + \sqrt{n_0 h_n}\left\{\widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}_0, z) - \boldsymbol{\delta}_0(\boldsymbol{\theta}_0, z)\right\}.
\end{aligned}$$

Assumption (C4) and $\widehat{\boldsymbol{\theta}}_n = O_p(1/\sqrt{n_0})$ imply that $\widehat{\boldsymbol{\delta}}_n(\widehat{\boldsymbol{\theta}}_n, z)$ and $\widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}_0, z)$ have the same asymptotic property. Therefore, to derive the conclusion in Theorem 1, it suffices to prove that $\widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}_0, z)$ is asymptotically normal for given $\boldsymbol{\theta}$ and z. Specifically, we take the first-order Taylor expansion of (6) on $\boldsymbol{\delta}_0$ and get

$$\begin{aligned}
&\sum_{i=1}^{n} K_{h_n}(Z_i - z)\frac{\partial}{\partial \boldsymbol{\delta}}\left[\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\boldsymbol{\delta}) - \mathbf{X}_i^\top\boldsymbol{\theta} - \bar{\mathbf{Z}}_i^\top\boldsymbol{\delta}\right] I(Y_i > w_n) \\
=&\sum_{i=1}^{n} K_{h_n}(Z_i - z)\left\{\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\boldsymbol{\delta}) - 1\right\}\bar{\mathbf{Z}}_i I(Y_i > w_n) \\
=&\sum_{i=1}^{n} K_{h_n}(Z_i - z)\left\{\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\boldsymbol{\delta}_0) - 1\right\}\bar{\mathbf{Z}}_i I(Y_i > w_n) \\
&+\sum_{i=1}^{n} K_{h_n}(Z_i - z)\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\tilde{\boldsymbol{\delta}})\bar{\mathbf{Z}}_i\bar{\mathbf{Z}}_i^\top I(Y_i > w_n)(\boldsymbol{\delta} - \boldsymbol{\delta}_0) = 0,
\end{aligned}$$

in which $\tilde{\boldsymbol{\delta}}$ is between $\boldsymbol{\delta}_0$ and $\widehat{\boldsymbol{\delta}}_n$ for given z and satisfies $\tilde{\boldsymbol{\delta}} \to \boldsymbol{\delta}_0$ in probability. Consequently,

$$\begin{aligned}
\widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}, z) - \boldsymbol{\delta}_0(\boldsymbol{\theta}_0, z) =& -\left\{\sum_{i=1}^{n} K_{h_n}(Z_i - z)\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\tilde{\boldsymbol{\delta}})\bar{\mathbf{Z}}_i\bar{\mathbf{Z}}_i^\top I(Y_i > w_n)\right\}^{-1} \\
&\cdot\sum_{i=1}^{n} K_{h_n}(Z_i - z)\left\{\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\boldsymbol{\delta}_0) - 1\right\}\bar{\mathbf{Z}}_i I(Y_i > w_n).
\end{aligned}$$

For ease of notation, we write

$$\mathbf{A}_n = \frac{1}{n_0}\sum_{i=1}^{n} K_{h_n}(Z_i - z)\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\tilde{\boldsymbol{\delta}})\bar{\mathbf{Z}}_i\bar{\mathbf{Z}}_i^\top I(Y_i > w_n)$$

and

$$\boldsymbol{\Sigma}_n = \sqrt{\frac{h_n}{n_0}}\sum_{i=1}^{n} K_{h_n}(Z_i - z)\left\{\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\boldsymbol{\delta}_0) - 1\right\}\bar{\mathbf{Z}}_i I(Y_i > w_n).$$

Now, we prove that $\mathbf{A}_n$ converges to $\mathbf{A}(z)$ in probability where $\mathbf{A}(z)$ is a diagonal matrix with entries

$$\begin{aligned}\mathbf{A}_{kk}(z) =& \frac{n}{n_0}\mu_{2(k-1)}\mathrm{E}\left\{c_0(\mathbf{X}_1, Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)} + c_1(\mathbf{X}_1, Z_1)\frac{\alpha(\mathbf{X}_1, Z_1)}{\alpha(\mathbf{X}_1, Z_1) + \beta(\mathbf{X}_1, Z_1)}\right.\\ &\left.\cdot w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}|Z_1 = z\right\} f_Z(z),\ \ k = 1, 2.\end{aligned}$$

Noting that the (k,l)th entry of $\mathbf{A}_n$ is

$$(\mathbf{A}_n)_{kl} = \frac{1}{n_0}\sum_{i=1}^{n} K_{h_n}(Z_i - z)\log(Y_i/w_n)\exp(\mathbf{X}_i^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_i^\top\tilde{\boldsymbol{\delta}})(\frac{Z_i - z}{h_n})^{k+l-2} I(Y_i > w_n), k, l = 1, 2,$$

then we just show that $(\mathbf{A}_n)_{kl} \to \mathbf{A}_{kl}$ in probability. Using that $e^x - 1 \approx x$ for $x \to 0$ and the argument in Section 5.6 of Fan and Gijbels (1996), we get

$$\begin{aligned}&\mathrm{E}\{(\mathbf{A}_n)_{kl}\} = \frac{n}{n_0}\mathrm{E}[K_{h_n}(Z_1 - z)\left(\frac{Z_1 - z}{h_n}\right)^{k+l-2}\exp\{\mathbf{X}_1^\top\boldsymbol{\theta} + \eta_0(Z_1)\}\\ &\quad\cdot \mathrm{E}\{\log(Y_1/w_n)I(Y_1 > w_n)|\mathbf{X}_1, Z_1\}] + O(\frac{nh_n^2}{2n_0})\\ &= \frac{n}{n_0}\mathrm{E}\left[K_{h_n}(Z_1 - z)\left(\frac{Z_1 - z}{h_n}\right)^{k+l-2}\exp\{\mathbf{X}_1^\top\boldsymbol{\theta} + \eta_0(Z_1)\}\int_0^\infty pr(Y_1 > w_n e^t)\,dt\right] + O(\frac{nh_n^2}{2n_0})\\ &= \frac{n}{n_0}\mathrm{E}\left[K_{h_n}(Z_1 - z)(\frac{Z_1 - z}{h_n})^{k+l-2}\left\{c_0(\mathbf{X}_1, Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}\right.\right.\\ &\quad\left.\left.+c_1(\mathbf{X}_1, Z_1)\frac{\alpha(\mathbf{X}_1, Z_1)}{\alpha(\mathbf{X}_1, Z_1) + \beta(\mathbf{X}_1, Z_1)}\cdot w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}\right] + O(\frac{nh_n^2}{2n_0})\\ &= \frac{n}{n_0}\mu_{k+l-2}\mathrm{E}\left\{c_0(\mathbf{X}_1, Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)} + c_1(\mathbf{X}_1, Z_1)\frac{\alpha(\mathbf{X}_1, Z_1)}{\alpha(\mathbf{X}_1, Z_1) + \beta(\mathbf{X}_1, Z_1)}\right.\\ &\quad\left.\cdot w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}|Z_1 = z\right\} f_Z(z) + o(1)\end{aligned}$$

moreover, the (k,l)th entry of the variance-covariance matrix of $\mathbf{A}_n$ is

$$\mathrm{var}\{(\mathbf{A}_n)_{kl}\} = \frac{n}{n_0^2}\mathrm{E}\left\{K_{h_n}(Z_1-z)\exp(\mathbf{X}_1^\top\boldsymbol{\theta}+\bar{\mathbf{Z}}_1^\top\tilde{\boldsymbol{\delta}})\log(Y_1/w_n)I(Y_1>w_n)(\frac{Z_1-z}{h_n})^{k+l-2}\right\}^2$$
$$-\frac{n}{n_0^2}\left[\mathrm{E}\left\{K_{h_n}(Z_1-z)\exp(\mathbf{X}_1^\top\boldsymbol{\theta}+\bar{\mathbf{Z}}_1^\top\tilde{\boldsymbol{\delta}})\log(Y_1/w_n)I(Y_1>w_n)(\frac{Z_1-z}{h_n})^{k+l-2}\right\}\right]^2$$
$$=\frac{n}{n_0^2h_n}\mathrm{E}\left[K^2((Z_1-z)/h_n)/h_n\exp\{2(\mathbf{X}_1^\top\boldsymbol{\theta}+\bar{\mathbf{Z}}_1^\top\boldsymbol{\delta}_0)\}\log^2(Y_1/w_n)I(Y_1>w_n)(\frac{Z_1-z}{h_n})^{2(k+l-2)}\right]$$
$$-O(\frac{1}{n_0})+O(\frac{nh_n}{n_0^2}).$$

Simple calculations and condition (C4) imply that $\mathrm{var}[(\mathbf{A}_n)_{kl}]=O(n/n_0^2h_n)+O(nh_n/n_0^2)=o(1)$. Therefore, $\mathbf{A}_n\to_p\mathbf{A}(z)$ in probability follows from the fact $R=\mathrm{E}(R)+O_p(\sqrt{\mathrm{var}(R)})$ for each random variable R and $\mathbf{A}(z)$ is diagonal that is guaranteed by the definition of μ_k,ν_k. Now, we just consider $(\boldsymbol{\Sigma}_n)_{11}$ that's the summation of independent and identically distributed variables. Thus, it suffices to construct asymptotic mean and variance for $\boldsymbol{\Sigma}_n$ by following CLT conditions. Specifically, the mean of the $(1,1)$th entry of $\boldsymbol{\Sigma}_n$ is

$$\mathrm{E}\{(\boldsymbol{\Sigma}_n)_{11}\}=\sqrt{\frac{n^2h_n}{n_0}}\mathrm{E}\left[K_{h_n}(Z_1-z)\{\log(Y_1/w_n)\exp(\mathbf{X}_1^\top\boldsymbol{\theta}+\bar{\mathbf{Z}}_1^\top\boldsymbol{\delta}_0)-1\}I(Y_1>w_n)\right]$$
$$=\sqrt{\frac{n^2h_n}{n_0}}\mathrm{E}\left\{K_{h_n}(Z_1-z)\exp(\mathbf{X}_1^\top\boldsymbol{\theta}+\bar{\mathbf{Z}}_1^\top\boldsymbol{\delta}_0)\int_0^\infty pr(Y_1>w_ne^t)\,dt\right\}$$
$$-\sqrt{\frac{n^2h_n}{n_0}}\mathrm{E}\{K_{h_n}(Z_1-z)pr(Y_1>w_n)\}$$
$$=\sqrt{\frac{n^2h_n}{n_0}}\mathrm{E}\left\{K_{h_n}(Z_1-z)\exp(\mathbf{X}_1^\top\boldsymbol{\theta}+\eta_0(Z_1))\left[1-\frac{h_n^2}{2}\eta^{(2)}(Z_1)\right]\int_0^\infty pr(Y_1>w_ne^t)\,dt\right\}$$
$$-\sqrt{\frac{n^2h_n}{n_0}}\mathrm{E}\{K_{h_n}(Z_1-z)pr(Y_1>w_n)\}$$
$$=-\sqrt{\frac{n^2h_n}{n_0}}\mathrm{E}\left\{\frac{h_n^2}{2}\eta^{(2)}(Z_1)\mu_2\left[c_0(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}+\frac{c_1(\mathbf{X}_1,Z_1)\alpha(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right]\right.$$
$$\left.-c_1(\mathbf{X}_1,Z_1)\frac{\beta(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}|Z_1=z\right\}f_Z(z)\to\boldsymbol{\Sigma}_{11}(z),$$

and the corresponding variance-covariance matrix is

$$
\begin{aligned}
&\mathrm{var}\{(\boldsymbol{\Sigma}_n)_{11}\} = \frac{nh_n}{n_0}\mathrm{var}\left[K_{h_n}(Z_1 - z)\left\{\log(Y_1/w_n)\exp(\mathbf{X}_1^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_1^\top\boldsymbol{\delta}_0) - 1\right\} I(Y_1 > w_n)\right] \\
&= \frac{nh_n}{n_0}\mathrm{E}\left[K_{h_n}^2(Z_1 - z)\left\{\log(Y_1/w_n)\exp(\mathbf{X}_1^\top\boldsymbol{\theta} + \bar{\mathbf{Z}}_1^\top\boldsymbol{\delta}_0) - 1\right\}^2 I(Y_1 > w_n)\right] + O(1/n) \\
&= \frac{n}{n_0}\mathrm{E}\left\{c_0(\mathbf{X}_1, Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)} + c_1(\mathbf{X}_1, Z_1)\frac{\alpha^2(\mathbf{X}_1, Z_1) + \beta^2(\mathbf{X}_1, Z_1)}{(\alpha(\mathbf{X}_1, Z_1) + \beta(\mathbf{X}_1, Z_1))^2}\right. \\
&\left.\cdot w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}|Z_1 = z\right\} f_Z(z)\nu_0 + o(1) \to \boldsymbol{\Lambda}_{11}(z).
\end{aligned}
$$

Thus, we conclude that

$$
\sqrt{n_0 h_n}\,(\widehat{\eta}_n(\boldsymbol{\theta}, z) - \eta_0(z)) = -(\mathbf{A}_{11}(z))^{-1}\boldsymbol{\Sigma}_{11}(z) + o_p(\frac{1}{\sqrt{n_0 h_n}} + h_n^2),
$$

which leads to the asymptotic variance matrix $[\mathbf{A}_{11}(z)]^{-1}\boldsymbol{\Lambda}_{11}(z)[\mathbf{A}_{11}(z)]^{-1}$. Therefore, we complete the proof.

Proof of Theorem 2. From the proof of Theorem 1, we can obtain that $\widehat{\eta}_n(\widehat{\boldsymbol{\theta}}_n, z) - \eta_0(z) = O_p(n_0^{-2/5})$ by taking $h_n = n_0^{-1/5}$ and $\widehat{\eta}_n(\boldsymbol{\theta}, z) - \eta_0(z) = O_p(n_0^{-2/5})$. Moreover, an under-smoothing condition $nh_n^4 \to 0$ is also needed in deriving asymptotic distribution of $\widehat{\boldsymbol{\theta}}$. We use $\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, z)$ and $\widehat{\eta}_n^{(2)}(\boldsymbol{\theta}_0, z)$ to denote the first and second order derivatives of $\widehat{\eta}_n(\boldsymbol{\theta}, z)$ at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. Then, the first-order Taylor expansion for (7) at $\boldsymbol{\theta}_0$ leads to

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{n_0}}\sum_{i=1}^n \frac{\partial}{\partial\boldsymbol{\theta}}\left[\log(Y_i/w_n)\exp\{\mathbf{X}_i^\top\boldsymbol{\theta} + \widehat{\eta}_n(\widehat{\boldsymbol{\theta}}_n, Z_i)\} - \{\mathbf{X}_i^\top\boldsymbol{\theta} + \widehat{\eta}_n(\widehat{\boldsymbol{\theta}}_n, Z_i)\}\right] I(Y_i > w_n) \\
&= \frac{1}{\sqrt{n_0}}\sum_{i=1}^n \frac{\partial}{\partial\boldsymbol{\theta}}\left[\log(Y_i/w_n)\exp\{\mathbf{X}_i^\top\boldsymbol{\theta} + \widehat{\eta}_n(\boldsymbol{\theta}, Z_i)\}\exp\{\widehat{\eta}_n(\widehat{\boldsymbol{\theta}}_n, Z_i) - \widehat{\eta}_n(\boldsymbol{\theta}, Z_i)\}\right. \\
&\quad \left. -\{\mathbf{X}_i^\top\boldsymbol{\theta} + \widehat{\eta}_n(\boldsymbol{\theta}, Z_i)\} - \{\widehat{\eta}_n(\widehat{\boldsymbol{\theta}}_n, Z_i) - \widehat{\eta}_n(\boldsymbol{\theta}, Z_i)\}\right] I(Y_i > w_n) \\
&= \frac{1}{\sqrt{n_0}}\sum_{i=1}^n \left[\log(Y_i/w_n)\exp\{\mathbf{X}_i^\top\boldsymbol{\theta}_0 + \widehat{\eta}_n(\boldsymbol{\theta}_0, Z_i)\} - 1\right]\{\mathbf{X}_i + \widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_i)\}I(Y_i > w_n) \\
&\quad + \left[\frac{1}{n_0}\sum_{i=1}^n \log(Y_i/w_n)I(Y_i > w_n)\exp\{\mathbf{X}_i^\top\boldsymbol{\theta}_0 + \widehat{\eta}_n(\boldsymbol{\theta}_0, Z_i)\}\{\mathbf{X}_i + \widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_i)\}\{\mathbf{X}_i + \widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_i)\}^\top\right. \\
&\quad \left. + \frac{1}{n_0}\sum_{i=1}^n \left[\log(Y_i/w_n)\exp\{\mathbf{X}_i^\top\boldsymbol{\theta}_0 + \widehat{\eta}_n(\boldsymbol{\theta}_0, Z_i)\} - 1\right]\widehat{\eta}_n^{(2)}(\boldsymbol{\theta}_0, Z_i)I(Y_i > w_n)\right]\sqrt{n_0}(\boldsymbol{\theta} - \boldsymbol{\theta}_0) + o_p(1) \\
&= \boldsymbol{\zeta}_n + (\boldsymbol{\Pi}_{1n} + \boldsymbol{\Pi}_{2n})\sqrt{n_0}(\boldsymbol{\theta} - \boldsymbol{\theta}_0) + o_p(1),
\end{aligned}
$$

which obviously implies that

$$
\sqrt{n_0}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) = -(\boldsymbol{\Pi}_{1n} + \boldsymbol{\Pi}_{2n})^{-1}\boldsymbol{\zeta}_n + o_p(1).
$$

Following the same argument used in proving Theorem 1, we get

$$\begin{aligned}
\mathrm{E}(\mathbf{\Pi}_{1n}) &= \frac{n}{n_0}\mathrm{E}\left[\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1)\}\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}^\top\right.\\
&\quad\left.\cdot\mathrm{E}\{\log(Y_1/w_n)I(Y_1>w_n)|\mathbf{X}_1,Z_1\}\right]\\
&= \frac{n}{n_0}\mathrm{E}\left[\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1)\}\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}^\top\int_0^\infty pr(Y_1>w_ne^t)\,dt\right]\\
&= \frac{n}{n_0}\mathrm{E}\left[\exp\left\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\bar{\mathbf{Z}}_1^\top\boldsymbol{\delta}(\boldsymbol{\theta}_0,z)+\bar{\mathbf{Z}}_1^\top(\widehat{\boldsymbol{\delta}}_n(\boldsymbol{\theta}_0,z)-\boldsymbol{\delta}(\boldsymbol{\theta}_0,z))\right\}\left\{\mathbf{X}_1+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1)+\bar{\mathbf{Z}}_1^\top(\widehat{\boldsymbol{\delta}}_n^{(1)}(\boldsymbol{\theta}_0,z)\right.\right.\\
&\quad\left.-\boldsymbol{\delta}^{(1)}(\boldsymbol{\theta}_0,z))\right\}\left\{\mathbf{X}_1+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1)+\bar{\mathbf{Z}}_1^\top(\widehat{\boldsymbol{\delta}}_n^{(1)}(\boldsymbol{\theta}_0,z)-\boldsymbol{\delta}^{(1)}(\boldsymbol{\theta}_0,z))\right\}^\top\left.\int_0^\infty pr(Y_1>w_ne^t)\,dt\right]\\
&= \frac{n}{n_0}\mathrm{E}\left[\exp\left\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\eta(\boldsymbol{\theta}_0,Z_1)\right\}\left\{\mathbf{X}_1+\eta_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\right\}\left\{\mathbf{X}_1+\eta_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\right\}^\top\right.\\
&\quad\left.\int_0^\infty pr(Y_1>w_ne^t)\,dt\right]+O(\frac{nh_n^2}{n_0})\\
&= \frac{n}{n_0}\mathrm{E}\left[\{\mathbf{X}_1+\eta_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\{\mathbf{X}_1+\eta_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}^\top\left\{c_0(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}\right.\right.\\
&\quad\left.\left.+c_1(\mathbf{X}_1,Z_1)\frac{\alpha(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}\right]+o(1)\quad\text{and}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(\mathbf{\Pi}_{2n}) &= \frac{n}{n_0}\mathrm{E}\left[\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\eta(\boldsymbol{\theta}_0,Z_1)\}\widehat{\eta}_n^{(2)}(\boldsymbol{\theta}_0,Z_1)\int_0^\infty pr(Y_1>w_ne^t)\,dt\right]\{1+O(nh_n^2/n_0)\}\\
&\quad-\frac{n}{n_0}\mathrm{E}\left[(\widehat{\eta}_n^{(2)}(\boldsymbol{\theta}_0,Z_1)pr(Y_1>w_n)\right]\\
&= \frac{n}{n_0}\mathrm{E}\left[\eta_n^{(2)}(\boldsymbol{\theta}_0,Z_1)\left\{c_0(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}+\frac{c_1(\mathbf{X}_1,Z_1)\alpha(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}\right]\\
&\quad-\frac{n}{n_0}\mathrm{E}\left[\eta_n^{(2)}(\boldsymbol{\theta}_0,Z_1)\left\{c_0(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}+c_1(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}\right]+o(1)\\
&= -\frac{n}{n_0}\mathrm{E}\left\{\eta_n^{(2)}(\boldsymbol{\theta}_0,Z_1)c_1(\mathbf{X}_1,Z_1)\frac{\beta(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}+o(1).
\end{aligned}$$

Similar discussions as those used in proving Theorem 1 lead to $\mathrm{cov}(\mathbf{\Pi}_{1n})=o(1)$ and $\mathrm{cov}(\mathbf{\Pi}_{2n})=o(1)$, which indicates that $\mathbf{\Pi}_{1n}\to\mathbf{\Pi}_1$ and $\mathbf{\Pi}_{2n}\to\mathbf{\Pi}_2$ in probability. To apply the Central Limiting Theorem condition to $\boldsymbol{\zeta}_n$, we now show that

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\zeta}_n) &= \frac{n}{\sqrt{n_0}}\mathrm{E}\left[\log(Y_1/w_n)\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1)\}-1\right]\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}I(Y_1>w_n)\\
&= \frac{n}{\sqrt{n_0}}\mathrm{E}\left[\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\eta(Z_1)\}\left\{1+\frac{h_n^2}{2}\eta^{(2)}(Z_1)+o(h_n^2)\right\}\right.\\
&\quad\left.\cdot\int_0^\infty pr(Y_1>w_ne^t)\,dt\right]-\frac{n}{\sqrt{n_0}}\mathrm{E}\left[\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}pr(Y_1>w_n)\right]+o(1)\\
&= -\frac{n}{\sqrt{n_0}}\mathrm{E}\left[\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\frac{c_1(\mathbf{X}_1,Z_1)\beta(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right]\\
&\quad-\frac{n}{\sqrt{n_0}}\frac{h_n^2}{2}\mathrm{E}\left[\eta^{(2)}(Z_1)\left\{c_0(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}+\frac{c_1(\mathbf{X}_1,Z_1)\alpha(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}\right.\\
&\quad\left.\cdot\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\right]+o(1)\\
&\to\boldsymbol{\zeta}_1+\boldsymbol{\zeta}_2,
\end{aligned}$$

where

$$\boldsymbol{\zeta}_1=\lim_{n_0\to\infty}-\frac{n}{\sqrt{n_0}}\mathrm{E}\left[\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\frac{c_1(\mathbf{X}_1,Z_1)\beta(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right],$$

and

$$\begin{aligned}
\boldsymbol{\zeta}_2=\lim_{n_0\to\infty}&-\frac{n}{\sqrt{n_0}}\frac{h_n^2}{2}\mathrm{E}\left[\eta^{(2)}(Z_1)\left\{c_0(\mathbf{X}_1,Z_1)w_n^{-\alpha(\mathbf{X}_1,Z_1)}+\frac{c_1(\mathbf{X}_1,Z_1)\alpha(\mathbf{X}_1,Z_1)}{\alpha(\mathbf{X}_1,Z_1)+\beta(\mathbf{X}_1,Z_1)}\right.\right.\\
&\left.\left.\cdot w_n^{-\alpha(\mathbf{X}_1,Z_1)-\beta(\mathbf{X}_1,Z_1)}\right\}\left\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\right\}\right].
\end{aligned}$$

Further, the variance-covariance matrix of $\boldsymbol{\zeta}_n$ is

$$\begin{aligned}
\mathrm{cov}(\boldsymbol{\zeta}_n) &= \frac{n}{n_0}\mathrm{cov}\left[\left\{\log(Y_1/w_n)\exp(\mathbf{X}_1^\top\boldsymbol{\theta}_0+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1))-1\right\}\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}I(Y_1>w_n)\right]\\
&= \frac{n}{n_0}\mathrm{E}\left[\log(Y_1/w_n)\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\widehat{\eta}_n(\boldsymbol{\theta}_0,Z_1)\}-1\right]^2\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}\\
&\quad\cdot\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\}^\top I(Y_1>w_n)-\frac{1}{n}\{\mathrm{E}(\boldsymbol{\zeta}_n)\}^2\\
&= \frac{n}{n_0}\mathrm{E}\left[\log(Y_1/w_n)\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0+\eta_0(Z_1)\}\left\{1+\frac{h_n^2}{2}\eta^{(2)}(Z_1)+o(h_n^2)\right\}-1\right]^2 I(Y_1>w_n)\\
&\quad\cdot\left\{\mathbf{X}_1\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\right\}\left\{\mathbf{X}_1+\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0,Z_1)\right\}^\top-\frac{1}{n}\{\mathrm{E}(\boldsymbol{\zeta}_n)\}^2.
\end{aligned}$$

Note that the combination of $n\boldsymbol{\zeta}_n/\sqrt{n_0}\to\boldsymbol{\zeta}$ in probability as $n_0\to\infty$ and $nh_n^4\to 0$ lead to

$\{\mathrm{E}(\boldsymbol{\zeta}_n)\}^2/n \to 0$. Moreover, the assumption $nh_n^2/n_0 \to 0$ in (C4) implies that

$$\begin{aligned}\mathrm{cov}(\boldsymbol{\zeta}_n) =& \frac{n}{n_0}\mathrm{E}\left[\log(Y_1/w_n)\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0 + \eta_0(Z_1)\} - 1\right]^2 \{\mathbf{X}_1 + \widehat{\eta}^{(1)}(\boldsymbol{\theta}_0, Z_1)\} \\ & \cdot \{\mathbf{X}_1 + \widehat{\eta}^{(1)}(\boldsymbol{\theta}_0, Z_1)\}^\top I(Y_1 > w_n) + o(1) \\ =& \frac{n}{n_0} pr(Y_1 > w_n)\mathrm{E}\left(\left[\log(Y_1/w_n)\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0 + \eta_0(Z_1)\} - 1\right]^2 \{\mathbf{X}_1 + \widehat{\eta}^{(1)}(\boldsymbol{\theta}_0, Z_1)\}\right. \\ & \left. \cdot\{\mathbf{X}_1 + \widehat{\eta}^{(1)}(\boldsymbol{\theta}_0, Z_1)\}^\top | Y_1 > w_n\right) + o(1).\end{aligned}$$

Just as Wang and Tsai (2009) showed, $npr(Y_1 > w_n)/n_0 \to 1$ and $\log(Y_1/w_n)\exp\{\mathbf{X}_1^\top\boldsymbol{\theta}_0 + \eta_0(Z_1)\}$ is distributed as a standard exponential distribution approximately. Finally, we consider the residual component:

$$\mathrm{E}\left[\{\mathbf{X}_1 + \widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_1)\}\{\mathbf{X}_1 + \widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_1)\}^\top | Y_1 > w_n\right] = \mathbf{V}_n,$$

then the application of condition (C8) to $\mathbf{V}_n$ leads to

$$\begin{aligned}\mathbf{V}_n =& \mathrm{E}\left[\left\{\mathbf{X}_1 + \eta^{(1)}(\boldsymbol{\theta}_0, Z_1) + \left(\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_1) - \eta^{(1)}(\boldsymbol{\theta}_0, Z_1)\right)\right\}\right. \\ & \left.\left\{\mathbf{X}_1 + \eta^{(1)}(\boldsymbol{\theta}_0, Z_1) + \left(\widehat{\eta}_n^{(1)}(\boldsymbol{\theta}_0, Z_1) - \eta^{(1)}(\boldsymbol{\theta}_0, Z_1)\right)\right\}^\top | Y_1 > w_n\right] \\ \to& \mathrm{E}\left[\{\mathbf{X}_1 + \eta^{(1)}(\boldsymbol{\theta}_0, Z_1)\}\{\mathbf{X}_1 + \eta^{(1)}(\boldsymbol{\theta}_0, Z_1)\}^\top | Y_1 > w_n\right] = \mathbf{V},\end{aligned}$$

Thus, we complete the proof of the first result. Further, the bias term $\boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2$ can be reduced in the proof of Theorem 2 when $nh_n^2/\sqrt{n_0} \to 0$ that can be derived once undersmoothing condition is satisfied.