# Supplementary Material to "Mean and Covariance Estimation for Functional Snippets"

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## S.1 Implementation

The proposed method has been implemented in the `R` package `mcfda`[1] for mean and covariance estimation in functional data analysis. To use the package, first apply the following command

```
devtools::install_github("linulysses/mcfda")
```

to install the package. For illustration, we use the following command

```
D <- synfd::sparse.fd(0, synfd::gaussian.process(), n=100, m=5, delta=0.5)
```

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[1]https://github.com/linulysses/mcfda.

from the `synfd`[2] package to synthesize a snippet sample of size $n = 100$ in which on average each snippet is observed at $m = 5$ random points on $[0, 1]$ and the span of each snippet is no larger than $\delta = 0.5$. Now call

```
cov.obj <- covfunc(D$t, D$y, method="SP")
```

with `method="SP"` to estimate the covariance function by the proposed method with a default setting in which Matérn correlation function is used. A customized correlation function instead of the default one can be adopted; see the package manual for details. Finally, call

```
cov.hat <- predict(cov.obj, seq(0,1,0.01))
```

to obtain the estimated covariance function in the grid $\{(s, t) : s, t = 0, 0.01, \ldots, 1\}$.

## S.2 Additional simulation results for $\sigma_0^2$

[2]https://github.com/linulysses/synfd. Installation of this package can be done by the command `devtools::install_github("linulysses/synfd")`.

Table S.1: RMSE and their standard errors for $\hat{\sigma}_0^2$ under the sparse design and $\mu_2$

| | | | method | | |
|---|---|---|---|---|---|
| Cov | $n$ | $\sigma_0^2$ | SNPT | PACE | LM |
| I | 50 | 0 | 0.011 (0.008) | 0.138 (0.150) | 0.146 (0.203) |
| | | 0.1 | 0.024 (0.030) | 0.144 (0.176) | 0.138 (0.145) |
| | | 0.25 | 0.062 (0.078) | 0.162 (0.173) | 0.129 (0.132) |
| | | 0.5 | 0.113 (0.145) | 0.173 (0.208) | 0.135 (0.132) |
| | 200 | 0 | 0.009 (0.005) | 0.080 (0.100) | 0.069 (0.076) |
| | | 0.1 | 0.014 (0.018) | 0.091 (0.098) | 0.144 (0.154) |
| | | 0.25 | 0.028 (0.033) | 0.083 (0.090) | 0.107 (0.105) |
| | | 0.5 | 0.052 (0.063) | 0.095 (0.107) | 0.139 (0.101) |
| II | 50 | 0 | 0.036 (0.029) | 0.273 (0.272) | 0.223 (0.247) |
| | | 0.1 | 0.043 (0.049) | 0.238 (0.230) | 0.236 (0.250) |
| | | 0.25 | 0.078 (0.107) | 0.246 (0.276) | 0.168 (0.193) |
| | | 0.5 | 0.124 (0.152) | 0.257 (0.279) | 0.114 (0.131) |
| | 200 | 0 | 0.024 (0.015) | 0.182 (0.186) | 0.171 (0.164) |
| | | 0.1 | 0.034 (0.035) | 0.190 (0.184) | 0.188 (0.193) |
| | | 0.25 | 0.044 (0.053) | 0.184 (0.179) | 0.143 (0.141) |
| | | 0.5 | 0.067 (0.078) | 0.170 (0.170) | 0.107 (0.081) |
| III | 50 | 0 | 0.003 (0.003) | 0.099 (0.107) | 0.014 (0.030) |
| | | 0.1 | 0.022 (0.027) | 0.098 (0.107) | 0.114 (0.152) |
| | | 0.25 | 0.058 (0.075) | 0.121 (0.130) | 0.105 (0.102) |
| | | 0.5 | 0.092 (0.121) | 0.109 (0.138) | 0.157 (0.127) |
| | 200 | 0 | 0.002 (0.001) | 0.063 (0.070) | 0.005 (0.009) |
| | | 0.1 | 0.010 (0.013) | 0.068 (0.067) | 0.064 (0.094) |
| | | 0.25 | 0.029 (0.035) | 0.070 (0.075) | 0.078 (0.073) |
| | | 0.5 | 0.053 (0.066) | 0.070 (0.081) | 0.148 (0.090) |

Table S.2: RMSE and their standard errors for $\hat{\sigma}_0^2$ under the dense design and $\mu_1$

| | | | method | | |
|---|---|---|---|---|---|
| Cov | $n$ | $\sigma_0^2$ | SNPT | PACE | LM |
| I | 50 | 0 | 0.013 (0.003) | 0.052 (0.053) | 0.036 (0.098) |
| | | 0.1 | 0.014 (0.012) | 0.050 (0.053) | 0.037 (0.052) |
| | | 0.25 | 0.019 (0.021) | 0.045 (0.044) | 0.039 (0.056) |
| | | 0.5 | 0.027 (0.032) | 0.043 (0.053) | 0.032 (0.034) |
| | 200 | 0 | 0.013 (0.002) | 0.043 (0.035) | 0.021 (0.008) |
| | | 0.1 | 0.013 (0.008) | 0.039 (0.031) | 0.024 (0.049) |
| | | 0.25 | 0.015 (0.013) | 0.036 (0.032) | 0.019 (0.025) |
| | | 0.5 | 0.018 (0.020) | 0.027 (0.027) | 0.015 (0.016) |
| II | 50 | 0 | 0.025 (0.009) | 0.172 (0.122) | 0.075 (0.043) |
| | | 0.1 | 0.025 (0.016) | 0.177 (0.130) | 0.086 (0.100) |
| | | 0.25 | 0.030 (0.029) | 0.161 (0.115) | 0.074 (0.061) |
| | | 0.5 | 0.037 (0.042) | 0.155 (0.127) | 0.060 (0.059) |
| | 200 | 0 | 0.026 (0.006) | 0.165 (0.084) | 0.091 (0.042) |
| | | 0.1 | 0.025 (0.012) | 0.159 (0.078) | 0.091 (0.043) |
| | | 0.25 | 0.027 (0.020) | 0.152 (0.079) | 0.092 (0.048) |
| | | 0.5 | 0.030 (0.028) | 0.141 (0.083) | 0.081 (0.053) |
| III | 50 | 0 | 0.003 (0.002) | 0.057 (0.048) | 0.002 (0.001) |
| | | 0.1 | 0.007 (0.007) | 0.051 (0.047) | 0.041 (0.078) |
| | | 0.25 | 0.014 (0.016) | 0.046 (0.047) | 0.034 (0.047) |
| | | 0.5 | 0.026 (0.033) | 0.051 (0.055) | 0.045 (0.043) |
| | 200 | 0 | 0.004 (0.002) | 0.053 (0.037) | 0.003 (0.002) |
| | | 0.1 | 0.006 (0.005) | 0.052 (0.034) | 0.004 (0.005) |
| | | 0.25 | 0.009 (0.010) | 0.049 (0.037) | 0.010 (0.011) |
| | | 0.5 | 0.014 (0.015) | 0.037 (0.032) | 0.022 (0.021) |

Table S.3: RMSE and their standard errors for $\hat{\sigma}_0^2$ under the dense design and $\mu_2$

| | | | method | | |
|---|---|---|---|---|---|
| Cov | $n$ | $\sigma_0^2$ | SNPT | PACE | LM |
| I | 50 | 0 | 0.012 (0.003) | 0.054 (0.050) | 0.019 (0.012) |
| | | 0.1 | 0.012 (0.011) | 0.048 (0.050) | 0.049 (0.056) |
| | | 0.25 | 0.017 (0.020) | 0.043 (0.044) | 0.058 (0.067) |
| | | 0.5 | 0.027 (0.033) | 0.043 (0.049) | 0.034 (0.039) |
| | 200 | 0 | 0.012 (0.003) | 0.045 (0.034) | 0.021 (0.008) |
| | | 0.1 | 0.012 (0.007) | 0.040 (0.033) | 0.031 (0.060) |
| | | 0.25 | 0.014 (0.013) | 0.034 (0.029) | 0.030 (0.045) |
| | | 0.5 | 0.017 (0.019) | 0.029 (0.031) | 0.016 (0.017) |
| II | 50 | 0 | 0.024 (0.008) | 0.168 (0.118) | 0.074 (0.042) |
| | | 0.1 | 0.025 (0.015) | 0.160 (0.123) | 0.082 (0.086) |
| | | 0.25 | 0.030 (0.027) | 0.161 (0.124) | 0.076 (0.057) |
| | | 0.5 | 0.037 (0.037) | 0.146 (0.114) | 0.059 (0.057) |
| | 200 | 0 | 0.025 (0.006) | 0.163 (0.084) | 0.090 (0.042) |
| | | 0.1 | 0.025 (0.012) | 0.161 (0.086) | 0.095 (0.043) |
| | | 0.25 | 0.026 (0.019) | 0.155 (0.084) | 0.092 (0.047) |
| | | 0.5 | 0.027 (0.027) | 0.142 (0.086) | 0.080 (0.049) |
| III | 50 | 0 | 0.002 (0.002) | 0.057 (0.049) | 0.001 (0.001) |
| | | 0.1 | 0.007 (0.007) | 0.053 (0.049) | 0.045 (0.054) |
| | | 0.25 | 0.013 (0.014) | 0.049 (0.050) | 0.042 (0.063) |
| | | 0.5 | 0.025 (0.031) | 0.050 (0.054) | 0.041 (0.041) |
| | 200 | 0 | 0.003 (0.002) | 0.055 (0.034) | 0.003 (0.003) |
| | | 0.1 | 0.005 (0.005) | 0.054 (0.037) | 0.004 (0.005) |
| | | 0.25 | 0.006 (0.007) | 0.046 (0.034) | 0.008 (0.010) |
| | | 0.5 | 0.012 (0.015) | 0.036 (0.034) | 0.023 (0.021) |

## S.3 Additional simulation results for $\sigma_X^2(t)$

Table S.4: RMISE and their standard errors for $\hat{\sigma}_X^2(t)$ under the sparse design and $\mu_2$

| | | | method | | |
|---|---|---|---|---|---|
| Cov | SNR | $n$ | SNPTM | PFBE | PACE |
| I | 2 | 50 | 0.523 (0.206) | 0.513 (0.222) | 2.012 (1.296) |
| | | 200 | 0.330 (0.118) | 0.319 (0.114) | 1.274 (0.880) |
| | 4 | 50 | 0.517 (0.219) | 0.480 (0.230) | 1.653 (1.261) |
| | | 200 | 0.315 (0.137) | 0.321 (0.136) | 1.213 (0.798) |
| II | 2 | 50 | 0.759 (0.281) | 0.731 (0.211) | 2.521 (1.546) |
| | | 200 | 0.495 (0.165) | 0.520 (0.156) | 1.890 (1.382) |
| | 4 | 50 | 0.756 (0.261) | 0.726 (0.246) | 2.236 (1.511) |
| | | 200 | 0.468 (0.178) | 0.492 (0.142) | 1.376 (0.982) |
| III | 2 | 50 | 0.580 (0.178) | 0.555 (0.138) | 1.598 (1.181) |
| | | 200 | 0.399 (0.234) | 0.412 (0.120) | 0.995 (0.581) |
| | 4 | 50 | 0.550 (0.183) | 0.550 (0.126) | 1.089 (0.885) |
| | | 200 | 0.354 (0.206) | 0.386 (0.145) | 0.953 (0.555) |

Table S.5: RMISE and their standard errors for $\hat{\sigma}_X^2(t)$ under the dense design and $\mu_1$

| | | | method | | |
|---|---|---|---|---|---|
| Cov | SNR | $n$ | SNPTM | PFBE | PACE |
| I | 2 | 50 | 0.488 (0.117) | 0.509 (0.245) | 0.588 (0.246) |
| | | 200 | 0.274 (0.082) | 0.275 (0.087) | 0.344 (0.109) |
| | 4 | 50 | 0.480 (0.115) | 0.502 (0.198) | 0.561 (0.221) |
| | | 200 | 0.264 (0.071) | 0.266 (0.077) | 0.331 (0.094) |
| II | 2 | 50 | 0.665 (0.157) | 0.676 (0.202) | 0.757 (0.254) |
| | | 200 | 0.393 (0.139) | 0.414 (0.108) | 0.526 (0.125) |
| | 4 | 50 | 0.657 (0.147) | 0.649 (0.214) | 0.747 (0.308) |
| | | 200 | 0.362 (0.118) | 0.396 (0.120) | 0.493 (0.120) |
| III | 2 | 50 | 0.504 (0.125) | 0.490 (0.960) | 0.793 (0.311) |
| | | 200 | 0.297 (0.096) | 0.238 (0.074) | 0.797 (0.184) |
| | 4 | 50 | 0.497 (0.125) | 0.414 (0.235) | 0.797 (0.297) |
| | | 200 | 0.271 (0.076) | 0.223 (0.070) | 0.786 (0.170) |

Table S.6: RMISE and their standard errors for $\hat{\sigma}_X^2(t)$ under the dense design and $\mu_2$

| | | | method | | |
|---|---|---|---|---|---|
| Cov | SNR | $n$ | SNPTM | PFBE | PACE |
| I | 2 | 50 | 0.491 (0.158) | 0.521 (0.150) | 0.573 (0.248) |
| | | 200 | 0.260 (0.083) | 0.259 (0.087) | 0.330 (0.101) |
| | 4 | 50 | 0.476 (0.132) | 0.511 (0.146) | 0.551 (0.259) |
| | | 200 | 0.248 (0.064) | 0.257 (0.089) | 0.322 (0.107) |
| II | 2 | 50 | 0.676 (0.159) | 0.668 (0.193) | 0.748 (0.311) |
| | | 200 | 0.390 (0.126) | 0.412 (0.109) | 0.524 (0.134) |
| | 4 | 50 | 0.654 (0.201) | 0.660 (0.156) | 0.726 (0.259) |
| | | 200 | 0.371 (0.105) | 0.396 (0.110) | 0.512 (0.124) |
| III | 2 | 50 | 0.505 (0.126) | 0.414 (0.201) | 0.839 (0.377) |
| | | 200 | 0.290 (0.113) | 0.274 (0.153) | 0.812 (0.215) |
| | 4 | 50 | 0.487 (0.149) | 0.398 (0.200) | 0.825 (0.367) |
| | | 200 | 0.275 (0.077) | 0.245 (0.127) | 0.787 (0.171) |

## S.4 Additional simulation results for $\mathcal{C}$

Table S.7: RMISE and their standard errors for $\hat{\mathcal{C}}$ under the sparse design and $\mu_2$

| | | | method | | | |
|---|---|---|---|---|---|---|
| Cov | SNR | $n$ | SNPTM | SNPTF | PFBE | PACE |
| I | 2 | 50 | 0.338 (0.111) | 0.443 (0.148) | 0.488 (0.169) | 1.416 (0.673) |
| | | 200 | 0.237 (0.089) | 0.354 (0.087) | 0.296 (0.082) | 1.015 (0.506) |
| | 4 | 50 | 0.319 (0.137) | 0.418 (0.133) | 0.421 (0.162) | 1.269 (0.684) |
| | | 200 | 0.221 (0.087) | 0.337 (0.084) | 0.271 (0.110) | 0.966 (0.409) |
| II | 2 | 50 | 0.567 (0.124) | 0.519 (0.174) | 0.545 (0.163) | 2.259 (1.290) |
| | | 200 | 0.481 (0.074) | 0.428 (0.132) | 0.468 (0.114) | 1.706 (0.747) |
| | 4 | 50 | 0.542 (0.129) | 0.463 (0.137) | 0.510 (0.176) | 1.973 (1.167) |
| | | 200 | 0.466 (0.059) | 0.413 (0.110) | 0.432 (0.098) | 1.471 (0.607) |
| III | 2 | 50 | 0.498 (0.085) | 0.492 (0.124) | 0.483 (0.104) | 1.305 (0.721) |
| | | 200 | 0.479 (0.049) | 0.446 (0.092) | 0.371 (0.061) | 1.137 (0.404) |
| | 4 | 50 | 0.485 (0.073) | 0.479 (0.122) | 0.477 (0.104) | 1.217 (0.603) |
| | | 200 | 0.473 (0.041) | 0.433 (0.083) | 0.363 (0.080) | 1.070 (0.373) |

Table S.8: RMISE and their standard errors for $\hat{\mathcal{C}}$ under the dense design and $\mu_1$

| | | | method | | | |
|---|---|---|---|---|---|---|
| Cov | SNR | $n$ | SNPTM | SNPTF | PFBE | PACE |
| I | 2 | 50 | 0.308 (0.096) | 0.412 (0.101) | 0.434 (0.132) | 0.582 (0.171) |
| | | 200 | 0.177 (0.064) | 0.301 (0.064) | 0.260 (0.077) | 0.469 (0.079) |
| | 4 | 50 | 0.289 (0.089) | 0.402 (0.079) | 0.423 (0.117) | 0.542 (0.128) |
| | | 200 | 0.169 (0.052) | 0.284 (0.067) | 0.250 (0.068) | 0.449 (0.078) |
| II | 2 | 50 | 0.528 (0.085) | 0.489 (0.116) | 0.516 (0.154) | 1.069 (0.324) |
| | | 200 | 0.397 (0.042) | 0.351 (0.085) | 0.382 (0.148) | 1.000 (0.213) |
| | 4 | 50 | 0.502 (0.090) | 0.465 (0.119) | 0.499 (0.175) | 1.045 (0.277) |
| | | 200 | 0.382 (0.035) | 0.341 (0.074) | 0.371 (0.134) | 0.981 (0.207) |
| III | 2 | 50 | 0.471 (0.044) | 0.453 (0.081) | 0.454 (0.146) | 0.949 (0.217) |
| | | 200 | 0.467 (0.027) | 0.427 (0.052) | 0.336 (0.046) | 1.015 (0.130) |
| | 4 | 50 | 0.463 (0.039) | 0.422 (0.085) | 0.393 (0.127) | 0.961 (0.221) |
| | | 200 | 0.463 (0.026) | 0.402 (0.046) | 0.337 (0.039) | 1.010 (0.130) |

Table S.9: RMISE and their standard errors for $\hat{\mathcal{C}}$ under the dense design and $\mu_2$

| | | | method | | | |
|---|---|---|---|---|---|---|
| Cov | SNR | $n$ | SNPTM | SNPTF | PFBE | PACE |
| I | 2 | 50 | 0.305 (0.104) | 0.405 (0.100) | 0.415 (0.140) | 0.590 (0.146) |
| | | 200 | 0.173 (0.062) | 0.295 (0.070) | 0.252 (0.072) | 0.477 (0.081) |
| | 4 | 50 | 0.294 (0.092) | 0.401 (0.098) | 0.400 (0.138) | 0.570 (0.159) |
| | | 200 | 0.163 (0.046) | 0.278 (0.054) | 0.238 (0.067) | 0.449 (0.082) |
| II | 2 | 50 | 0.533 (0.086) | 0.519 (0.113) | 0.526 (0.159) | 1.048 (0.305) |
| | | 200 | 0.420 (0.041) | 0.352 (0.071) | 0.393 (0.154) | 1.005 (0.233) |
| | 4 | 50 | 0.523 (0.092) | 0.489 (0.116) | 0.502 (0.186) | 1.023 (0.296) |
| | | 200 | 0.392 (0.033) | 0.346 (0.065) | 0.378 (0.147) | 0.970 (0.194) |
| III | 2 | 50 | 0.479 (0.047) | 0.446 (0.096) | 0.395 (0.089) | 0.977 (0.277) |
| | | 200 | 0.464 (0.027) | 0.420 (0.051) | 0.351 (0.152) | 0.926 (0.151) |
| | 4 | 50 | 0.482 (0.049) | 0.427 (0.090) | 0.393 (0.085) | 0.968 (0.245) |
| | | 200 | 0.461 (0.026) | 0.409 (0.049) | 0.337 (0.066) | 0.916 (0.116) |