Supplementary Material for "Estimation of Subgraph Densities in Noisy Networks" by Chang, Kolaczyk and Yao

Proof of Theorem 1

Recalling the definition of F_M and F_{M^*} as the joint distributions of $\mathbf{Y}$ when $\mathbf{Y}$ follows models M and M^*, respectively, denote by $F_{i,j,M}$ and F_{i,j,M^*} the corresponding marginal distribution of $Y_{i,j}$. From Assumption 1, we have

$$\mathcal{H}^2(F_M, F_{M^*}) \leq \sum_{(i,j)\in\mathcal{S}} \mathcal{H}^2(F_{i,j,M}, F_{i,j,M^*}) + \sum_{(i,j)\in\mathcal{S}^c} \mathcal{H}^2(F_{i,j,M}, F_{i,j,M^*}) \ ,$$

where $\mathcal{S} = \mathrm{supp}(\mathbf{A})$, $\mathcal{S}^c = \mathrm{supp}(\mathbf{A}^*)$, and $\mathcal{H}(\cdot,\cdot)$ denotes the Hellinger distance between two distributions. Since $F_{i,j,M} = F_{i,j,M^*}$ for any $i \neq j$ which implies $\mathcal{H}^2(F_{i,j,M}, F_{i,j,M^*}) = 0$, then $\mathcal{H}^2(F_M, F_{M^*}) = 0$.

Without lose of generality, we assume $d_f = |f(M) - f(M^*)|$ for some $M \in \mathcal{M}$ with $f(M) < f(M^*)$. For any $\hat{f} \in \mathcal{E}$, we consider the hypothesis testing problem $H_0 : \mathbf{Y} \sim M$ versus $H_1 : \mathbf{Y} \sim M^*$, and define the test function $\Psi = I\{\hat{f} > f(M) + d_f/2\}$, which means we reject H_0 if $\Psi = 1$ and accept H_0 if $\Psi = 0$. The testing affinity (Le Cam, 1973, 2012) is defined as

$$\pi = \inf_{\substack{0\leq\phi\leq 1\\ \phi\text{-measurable}}} \mathbb{E}_{H_0}(\phi) + \mathbb{E}_{H_1}(1-\phi),$$

and it is the minimal sum of type I and type II errors of any test between H_0 and H_1. Recall $\mathcal{H}(F_M, F_{M^*}) = 0$ and $\pi \geq 1 - \mathcal{H}(F_M, F_{M^*})$, then $\pi = 1$. Notice that $\mathbb{P}_M(|\hat{f} - f| \geq d_f/2) \geq \mathbb{P}_M(\hat{f} > f + d_f/2) =$ type I error and $\mathbb{P}_{M^*}(|\hat{f} - f| \geq d_f/2) \geq \mathbb{P}_{M^*}(\hat{f} \leq f - d_f/2) = \mathbb{P}_{M^*}\{\hat{f} \leq f(M) + d_f/2\} =$ type II error. Thus $\max\{\mathbb{P}_M(|\hat{f} - f| \geq$

$d_f/2), \mathbb{P}_{M^*}(|\hat{f} - f| \geq d_f/2)\} \geq 1/2$ which implies

$$\sup_{\mathcal{M}} \mathbb{P}\biggl(|\hat{f} - f| \geq \frac{d_f}{2}\biggr) \geq \frac{1}{2}\,.$$

Since the above result holds for any $\hat{f} \in \mathcal{E}$, the proof of Theorem 1 is complete. $\square$

A useful lemma

To prove Proposition 1 and Theorems 2 and 3, we need the following lemma.

Lemma 1. *Let $N = p(p-1)/2$, $\kappa_1 = \alpha(1-\alpha)$ and $\kappa_2 = \beta(1-\beta)$. Under Assumption 1, if $N_1 = p(p-1)\delta \to \infty$ and $N_2 = p(p-1)(1-\delta) \to \infty$, it holds that $\sqrt{N}(\hat{u}_1 - u_1, \hat{u}_2 - u_2, \hat{u}_3 - u_3)^{\mathrm{T}} \to_d \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = (\sigma_{ij})_{3\times 3}$, where $\sigma_{11} = \delta\kappa_2 + (1-\delta)\kappa_1$, $\sigma_{22} = \delta\kappa_2(1/2-\kappa_2)+(1-\delta)\kappa_1(1/2-\kappa_1)$, $\sigma_{33} = \delta\beta\kappa_2(1/3-\beta\kappa_2)+(1-\delta)\kappa_1(1-\alpha)\{1/3-\kappa_1(1-\alpha)\}$, $\sigma_{12} = \sigma_{21} = \delta\kappa_2(\beta - 1/2) + (1-\delta)\kappa_1(1/2-\alpha)$, $\sigma_{13} = \sigma_{31} = \delta\kappa_2(\beta^2/3 - 2\kappa_2/3) + (1-\delta)\kappa_1\{(1-\alpha)^2/3 - 2\kappa_1/3\}$ and $\sigma_{23} = \sigma_{32} = \delta\beta\kappa_2(1/3-\kappa_2) + (1-\delta)(1-\alpha)\kappa_1(1/3-\kappa_1)$.*

Proof. Let $\mathcal{S} = \{(i,j) : A_{i,j} = 1, i < j\}$ and $\mathcal{S}^c = \{(i,j) : A_{i,j} = 0, i < j\}$. By the definition of $\hat{u}_k$ and u_k ($k = 1,2,3$), we have

$$\begin{aligned}
\hat{u}_1 - u_1 &= \frac{1}{N}\sum_{(i,j)\in\mathcal{S}}\{Y_{i,j} - (1-\beta)\} + \frac{1}{N}\sum_{(i,j)\in\mathcal{S}^c}(Y_{i,j} - \alpha),\\
\hat{u}_2 - u_2 &= \frac{1}{2N}\sum_{(i,j)\in\mathcal{S}}(|Y_{i,j,*} - Y_{i,j}| - 2\kappa_2) + \frac{1}{2N}\sum_{(i,j)\in\mathcal{S}^c}(|Y_{i,j,*} - Y_{i,j}| - 2\kappa_1)\\
\hat{u}_3 - u_3 &= \frac{1}{3N}\sum_{(i,j)\in\mathcal{S}}(\xi_{i,j} - 3\beta\kappa_2) + \frac{1}{3N}\sum_{(i,j)\in\mathcal{S}^c}\{\xi_{i,j} - 3\kappa_1(1-\alpha)\}
\end{aligned}$$

where $\xi_{i,j} = I(Y_{i,j,**} - 2Y_{i,j,*} + Y_{i,j} = 1 \text{ or } -2)$. It follows from Assumption 1 that

$$\begin{aligned}
N\mathbb{E}\{(\hat{u}_1 - u_1)^2\} &= \delta\kappa_2 + (1-\delta)\kappa_1 = \sigma_{11}, \\
N\mathbb{E}\{(\hat{u}_2 - u_2)^2\} &= \delta\kappa_2\left(\frac{1}{2} - \kappa_2\right) + (1-\delta)\kappa_1\left(\frac{1}{2} - \kappa_1\right) = \sigma_{22}, \\
N\mathbb{E}\{(\hat{u}_3 - u_3)^2\} &= \delta\beta\kappa_2\left(\frac{1}{3} - \beta\kappa_2\right) + (1-\delta)\kappa_1(1-\alpha)\left\{\frac{1}{3} - \kappa_1(1-\alpha)\right\} = \sigma_{33}, \\
N\mathbb{E}\{(\hat{u}_1 - u_1)(\hat{u}_2 - u_2)\} &= \delta\kappa_2\left(\beta - \frac{1}{2}\right) + (1-\delta)\kappa_1\left(\frac{1}{2} - \alpha\right) = \sigma_{12}, \\
N\mathbb{E}\{(\hat{u}_1 - u_1)(\hat{u}_3 - u_3)\} &= \delta\kappa_2\left(\frac{\beta^2}{3} - \frac{2\kappa_2}{3}\right) + (1-\delta)\kappa_1\left\{\frac{(1-\alpha)^2}{3} - \frac{2\kappa_1}{3}\right\} = \sigma_{13}, \\
N\mathbb{E}\{(\hat{u}_2 - u_2)(\hat{u}_3 - u_3)\} &= \delta\beta\kappa_2\left(\frac{1}{3} - \kappa_2\right) + (1-\delta)(1-\alpha)\kappa_1\left(\frac{1}{3} - \kappa_1\right) = \sigma_{23}.
\end{aligned}$$

By the Lindberg-Feller Central Limit Theorem, we have Lemma 1. □

Proof of Proposition 1

Define $g_1(x,y,z) = (1-z)x + z(1-y)$ and $g_2(x,y,z) = (1-z)x(1-x) + zy(1-y)$ for any $(x,y,z) \in (0,1)^3$. When α is known, it holds that $g_1(\alpha,\hat{\beta},\hat{\delta}) - g_1(\alpha,\beta,\delta) = \hat{u}_1 - u_1$ and $g_2(\alpha,\hat{\beta},\hat{\delta}) - g_2(\alpha,\beta,\delta) = \hat{u}_2 - u_2$. Since the equations $g_1(\alpha,y,z) = u_1$ and $g_2(\alpha,y,z) = u_2$ have the unique solution $(y,z) = (\beta,\delta)$, and $(\hat{u}_1,\hat{u}_2) = (u_1,u_2) + o_p(1)$, we have consistency of $(\hat{\beta},\hat{\delta})$. By Taylor expansion, we have $\mathbf{D}_\alpha(\hat{\beta}-\beta,\hat{\delta}-\delta)^{\mathrm{T}} = (\hat{u}_1-u_1,\hat{u}_2-u_2)^{\mathrm{T}}$ with

$$\mathbf{D}_\alpha = \left.\begin{pmatrix} \frac{\partial g_1(x,y,z)}{\partial y} & \frac{\partial g_1(x,y,z)}{\partial z} \\ \frac{\partial g_2(x,y,z)}{\partial y} & \frac{\partial g_2(x,y,z)}{\partial z} \end{pmatrix}\right|_{(x,y,z)=(\alpha,\beta^*,\delta^*)} \tag{S.1}$$

where $(\beta^*,\delta^*) = \lambda\cdot(\beta,\delta) + (1-\lambda)\cdot(\hat{\beta},\hat{\delta})$ for some $\lambda \in (0,1)$. Notice that $\det(\mathbf{D}_\alpha) = -\delta^*(1-\alpha-\beta^*)^2$. Since $\delta(1-\alpha-\beta)^2 \geq c$ for some positive constant c, with the continuity of the function $\delta(1-\alpha-\beta)^2$ with respect to (β,δ), we know $\det(\mathbf{D}_\alpha) \leq -c/2$ with probability approaching one. Therefore, $(\hat{\beta}-\beta,\hat{\delta}-\delta)^{\mathrm{T}} = \mathbf{D}_\alpha^{-1}(\hat{u}_1-u_1,\hat{u}_2-u_2)^{\mathrm{T}}$.

From Lemma 1, $(\hat{u}_1 - u_1, \hat{u}_2 - u_2) = O_p(N^{-1/2})$ which implies part (i) of Proposition 1. Analogously, we have part (ii). $\square$

Proof of Theorem 2

It follows from Lemma 1 that $\sqrt{N}(\hat{u}_1 - u_1, \hat{u}_2 - u_2)^{\mathrm{T}} \to_d \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_1)$ where $\mathbf{\Sigma}_1 = (\sigma_{ij})_{2\times 2}$ with σ_{ij} specified in Lemma 1. We first consider the case with known α. As we have shown in the proof of Proposition 1, $(\hat{\beta} - \beta, \hat{\delta} - \delta)^{\mathrm{T}} = \mathbf{D}_\alpha^{-1}(\hat{u}_1 - u_1, \hat{u}_2 - u_2)^{\mathrm{T}}$ with

$$\mathbf{D}_\alpha^{-1} = -\frac{1}{\delta^*(1-\alpha-\beta^*)^2}\begin{pmatrix} \beta^*(1-\beta^*) - \alpha(1-\alpha) & -(1-\alpha-\beta^*) \\ -\delta^*(1-2\beta^*) & -\delta^* \end{pmatrix}.$$

Therefore, $\sqrt{N}(\hat{\beta} - \beta, \hat{\delta} - \delta)^{\mathrm{T}} \to_d \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{1,\alpha})$ with

$$\mathbf{\Sigma}_{1,\alpha} = \frac{1}{\delta^2\kappa_3^4}\begin{pmatrix} \kappa_2 - \kappa_1 & -\kappa_3 \\ -\delta(1-2\beta) & -\delta \end{pmatrix}\mathbf{\Sigma}_1\begin{pmatrix} \kappa_2 - \kappa_1 & -\delta(1-2\beta) \\ -\kappa_3 & -\delta \end{pmatrix}$$

where $\kappa_1 = \alpha(1-\alpha)$, $\kappa_2 = \beta(1-\beta)$ and $\kappa_3 = 1 - \alpha - \beta$. This completes part (i) of Theorem 2. For part (ii), notice that

$$\mathbf{D}_\beta = \begin{pmatrix} \frac{\partial g_1(x,y,z)}{\partial x} & \frac{\partial g_1(x,y,z)}{\partial z} \\ \frac{\partial g_2(x,y,z)}{\partial x} & \frac{\partial g_2(x,y,z)}{\partial z} \end{pmatrix}\Bigg|_{(x,y,z)=(\alpha^*,\beta,\delta^*)},$$

where $(\alpha^*, \delta^*) = \lambda \cdot (\alpha, \delta) + (1-\lambda)\cdot(\hat{\alpha}, \hat{\delta})$ for some $\lambda \in (0,1)$. Then

$$\mathbf{D}_\beta^{-1} = -\frac{1}{(1-\delta^*)(1-\alpha^*-\beta)^2}\begin{pmatrix} \beta(1-\beta) - \alpha^*(1-\alpha^*) & -(1-\alpha^*-\beta) \\ -(1-\delta^*)(1-2\alpha^*) & 1-\delta^* \end{pmatrix}.$$

Since $(\hat{\alpha}-\alpha,\hat{\delta}-\delta)^{\mathrm{T}} = \mathbf{D}_\beta^{-1}(\hat{u}_1-u_1,\hat{u}_2-u_2)^{\mathrm{T}}$, then $\sqrt{N}(\hat{\alpha}-\alpha,\hat{\delta}-\delta)^{\mathrm{T}} \to_d \mathcal{N}(\mathbf{0},\mathbf{\Sigma}_{1,\beta})$ with

$$\mathbf{\Sigma}_{1,\beta} = \frac{1}{(1-\delta)^2\kappa_3^4}\begin{pmatrix} \kappa_2-\kappa_1 & -\kappa_3 \\ -(1-\delta)(1-2\alpha) & 1-\delta \end{pmatrix}\mathbf{\Sigma}_1\begin{pmatrix} \kappa_2-\kappa_1 & -(1-\delta)(1-2\alpha) \\ -\kappa_3 & 1-\delta \end{pmatrix}.$$

Therefore, we have part (ii). □

Proof of Theorem 3

Define $g_3(x,y,z) = (1-z)x(1-x)^2 + zy^2(1-y)$ for any $(x,y,z)\in(0,1)^3$. Recall $g_1(x,y,z) = (1-z)x + z(1-y)$ and $g_2(x,y,z) = (1-z)x(1-x) + zy(1-y)$. Following the same arguments in the proof of Proposition 1 for the consistency of $(\hat{\beta},\hat{\delta})$, we have the consistency of $(\hat{\alpha},\hat{\beta},\hat{\delta})$. By Taylor expansion, we have $\mathbf{D}(\hat{\alpha}-\alpha,\hat{\beta}-\beta,\hat{\delta}-\delta)^{\mathrm{T}} = (\hat{u}_1-u_1,\hat{u}_2-u_2,\hat{u}_3-u_3)^{\mathrm{T}}$ with

$$\mathbf{D} = \begin{pmatrix} \frac{\partial g_1(x,y,z)}{\partial x} & \frac{\partial g_1(x,y,z)}{\partial y} & \frac{\partial g_1(x,y,z)}{\partial z} \\ \frac{\partial g_2(x,y,z)}{\partial x} & \frac{\partial g_2(x,y,z)}{\partial y} & \frac{\partial g_2(x,y,z)}{\partial z} \\ \frac{\partial g_3(x,y,z)}{\partial x} & \frac{\partial g_3(x,y,z)}{\partial y} & \frac{\partial g_3(x,y,z)}{\partial z} \end{pmatrix}\Bigg|_{(x,y,z)=(\alpha^*,\beta^*,\delta^*)},$$

where $(\alpha^*,\beta^*,\delta^*) = \lambda\cdot(\alpha,\beta,\delta) + (1-\lambda)\cdot(\hat{\alpha},\hat{\beta},\hat{\delta})$ for some $\lambda\in(0,1)$. Notice that $\det(\mathbf{D}) = -(1-\delta^*)\delta^*(1-\alpha^*-\beta^*)^4$. Since $(1-\delta)\delta(1-\alpha-\beta)^4 \geq c$ for some positive constant c, with the continuity of the function $(1-\delta)\delta(1-\alpha-\beta)^4$ with respect to (α,β,δ), we know $\det(\mathbf{D}) \leq -c/2$ with probability approaching one. Therefore, $(\hat{\alpha}-\alpha,\hat{\beta}-\beta,\hat{\delta}-\delta)^{\mathrm{T}} = \mathbf{D}^{-1}(\hat{u}_1-u_1,\hat{u}_2-u_2,\hat{u}_3-u_3)^{\mathrm{T}}$. From Lemma 1, $(\hat{u}_1-u_1,\hat{u}_2-u_2,\hat{u}_3-u_3) = O_p(N^{-1/2})$

which implies $(\hat{\alpha}-\alpha, \hat{\beta}-\beta, \hat{\delta}-\delta) = O_p(N^{-1/2})$. Since

$$\mathbf{D}^{-1} = \begin{pmatrix} \frac{(1-2\beta^*)\alpha^*+\beta^{*2}}{(1-\delta^*)(1-\alpha^*-\beta^*)^2} & \frac{\alpha^*-2\beta^*}{(1-\delta^*)(1-\alpha^*-\beta^*)^2} & \frac{1}{(1-\delta^*)(1-\alpha^*-\beta^*)^2} \\ -\frac{(1-2\alpha^*)\beta^*+\alpha^{*2}}{\delta^*(1-\alpha^*-\beta^*)^2} & \frac{\beta^*-2\alpha^*+1}{\delta^*(1-\alpha^*-\beta^*)^2} & -\frac{1}{\delta^*(1-\alpha^*-\beta^*)^2} \\ -\frac{3(\alpha^*+\beta^*)-6\alpha^*\beta^*-1}{(1-\alpha^*-\beta^*)^3} & -\frac{3\alpha^*-3\beta^*-1}{(1-\alpha^*-\beta^*)^3} & -\frac{2}{(1-\alpha^*-\beta^*)^3} \end{pmatrix},$$

then $\sqrt{N}(\hat{\alpha}-\alpha, \hat{\beta}-\beta, \hat{\delta}-\delta)^{\mathrm{T}} \to_d \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_2)$ with

$$\mathbf{\Sigma}_2 = \begin{pmatrix} \frac{(1-2\beta)\alpha+\beta^2}{(1-\delta)\kappa_3^2} & \frac{\alpha-2\beta}{(1-\delta)\kappa_3^2} & \frac{1}{(1-\delta)\kappa_3^2} \\ -\frac{(1-2\alpha)\beta+\alpha^2}{\delta\kappa_3^2} & \frac{\beta-2\alpha+1}{\delta\kappa_3^2} & -\frac{1}{\delta\kappa_3^2} \\ \frac{3\kappa_3+6\alpha\beta-2}{\kappa_3^3} & \frac{3\kappa_3+6\beta-2}{\kappa_3^3} & -\frac{2}{\kappa_3^3} \end{pmatrix} \mathbf{\Sigma} \begin{pmatrix} \frac{(1-2\beta)\alpha+\beta^2}{(1-\delta)\kappa_3^2} & -\frac{(1-2\alpha)\beta+\alpha^2}{\delta\kappa_3^2} & \frac{3\kappa_3+6\alpha\beta-2}{\kappa_3^3} \\ \frac{\alpha-2\beta}{(1-\delta)\kappa_3^2} & \frac{\beta-2\alpha+1}{\delta\kappa_3^2} & \frac{3\kappa_3+6\beta-2}{\kappa_3^3} \\ \frac{1}{(1-\delta)\kappa_3^2} & -\frac{1}{\delta\kappa_3^2} & -\frac{2}{\kappa_3^3} \end{pmatrix},$$

where $\kappa_1 = \alpha(1-\alpha)$, $\kappa_2 = \beta(1-\beta)$, $\kappa_3 = 1-\alpha-\beta$, and $\mathbf{\Sigma}$ is specified in Lemma 1. This completes the proof of Theorem 3. $\square$

Proof of Proposition 2

Define

$$T_{\mathcal{V}}(\tau_1, \ldots, \tau_k) = \frac{1}{|\mathcal{V}|} \sum_{\mathbf{v}=(i_1,i_1',\ldots,i_k,i_k')\in\mathcal{V}} \prod_{\ell=1}^{k} \mathbb{E}\{\varphi_\ell(\mathrm{Y}_{i_\ell,i_\ell'})\}\,.$$

Since $|1-\alpha-\beta| \geq c$ for some positive constant c, the convergence rate of $|\tilde{C}_{\mathcal{V}}(\tau_1, \ldots, \tau_k) - C_{\mathcal{V}}(\tau_1, \ldots, \tau_k)|$ is the same as that of $|\tilde{T}_{\mathcal{V}}(\tau_1, \ldots, \tau_k) - T_{\mathcal{V}}(\tau_1, \ldots, \tau_k)|$. To simplify the notation, we write $\tilde{T}_{\mathcal{V}}(\tau_1, \ldots, \tau_k)$ and $T_{\mathcal{V}}(\tau_1, \ldots, \tau_k)$ as $\tilde{T}_{\mathcal{V}}$ and $T_{\mathcal{V}}$, respectively. Let

$\mathring{\varphi}_\ell(Y_{i_\ell,i'_\ell}) = \varphi_\ell(Y_{i_\ell,i'_\ell}) - \mathbb{E}\{\varphi_\ell(Y_{i_\ell,i'_\ell})\}$. Notice that

$$\begin{aligned}
\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}} &= \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\prod_{\ell=1}^k \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) - \prod_{\ell=1}^k \mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg] \\
&= \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\sum_{\substack{\xi_1+\cdots+\xi_k=1 \\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^k \prod_{\ell=1}^k \mathring{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell} \\
&= \sum_{\substack{\xi_1+\cdots+\xi_k=1 \\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^k \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^k \mathring{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}.
\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\mathbb{E}\big(|\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}}|^2\big) \le J_k \sum_{\substack{\xi_1+\cdots+\xi_k=1 \\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^k \mathbb{E}\bigg\{\bigg(\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^k \mathring{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}\bigg)^2\bigg\}$$

where $J_k = 2^k - 1$. For any given $\xi_1, \ldots, \xi_k \in \{0, 1\}$, define

$$\psi_{\xi_1,\ldots,\xi_k}(\mathbf{v}) = \prod_{\ell=1}^k \mathring{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}$$

with $\mathbf{v} = (i_1, i'_1, \ldots, i_k, i'_k) \in \mathcal{V}$. Therefore,

$$\mathbb{E}\big(|\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}}|^2\big) \le J_k \sum_{\substack{\xi_1+\cdots+\xi_k=1 \\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^k \mathbb{E}\bigg\{\bigg(\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\psi_{\xi_1,\ldots,\xi_k}(\mathbf{v})\bigg)^2\bigg\}. \tag{S.2}$$

For $\aleph_{\mathcal{V}}(s)$ defined in (20), we adopt the convention $\aleph_{\mathcal{V}}(0) = 1$. If $\xi_1 + \cdots + \xi_k = s$ with $1 \le s \le k$, without lose of generality, we assume $\xi_1 = \cdots = \xi_s = 1$ and $\xi_{s+1} = \cdots = \xi_k = 0$. Then

$$\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\psi_{1,\ldots,1,0,\ldots,0}(\mathbf{v}) = \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\bigg(\prod_{\ell=1}^s \mathring{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\cdot\prod_{\ell=s+1}^k \mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg).$$

For any $\mathbf{v} = (i_1, i_1', \ldots, i_k, i_k') \in \mathcal{V}$ and $\tilde{\mathbf{v}} = (\tilde{i}_1, \tilde{i}_1', \ldots, \tilde{i}_k, \tilde{i}_k') \in \mathcal{V}$, if $|\{\{i_1, i_1'\}, \ldots, \{i_s, i_s'\}\} \cap \{\{\tilde{i}_1, \tilde{i}_1'\}, \ldots, \{\tilde{i}_s, \tilde{i}_s'\}\}| < s$, then $\mathbb{E}\{\psi_{1,\ldots,1,0,\ldots,0}(\mathbf{v})\psi_{1,\ldots,1,0,\ldots,0}(\tilde{\mathbf{v}})\} = 0$. Recall that $|\psi_{\xi_1,\ldots,\xi_k}(\mathbf{v})| \leq q_{\max}^k$ for any $\xi_1, \ldots, \xi_k \in \{0,1\}$ and $\mathbf{v} \in \mathcal{V}$, where $q_{\max} = \max\{1-\alpha, \alpha, 1-\beta, \beta\}$. Thus,

$$\mathbb{E}\bigg[\bigg\{\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}} \psi_{1,\ldots,1,0\ldots,0}(\mathbf{v})\bigg\}^2\bigg] \leq \frac{2^s s! \aleph_{\mathcal{V}}(k-s)}{|\mathcal{V}|^2} \sum_{\mathbf{v}\in\mathcal{V}} q_{\max}^{2k} = \frac{2^s s! q_{\max}^{2k} \aleph_{\mathcal{V}}(k-s)}{|\mathcal{V}|} .$$

Similarly, we know

$$\mathbb{E}\bigg[\bigg\{\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}} \psi_{\xi_1,\ldots,\xi_k}(\mathbf{v})\bigg\}^2\bigg] \leq \frac{2^s s! q_{\max}^{2k} \aleph_{\mathcal{V}}(k-s)}{|\mathcal{V}|} \tag{S.3}$$

for any $\xi_1, \ldots, \xi_k \in \{0,1\}$ such that $\xi_1 + \cdots + \xi_k = s$. Therefore, from (S.2), it holds that

$$\mathbb{E}\big(|\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}}|^2\big) \leq \frac{2^k k! q_{\max}^{2k} J_k^2}{|\mathcal{V}|} \max_{1\leq s\leq k} \aleph_{\mathcal{V}}(k-s) = \frac{2^k k! q_{\max}^{2k} J_k^2}{|\mathcal{V}|} \aleph_{\mathcal{V}} . \tag{S.4}$$

It follows from Markov inequality that

$$|\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}}| = O_p\bigg(\sqrt{\frac{\aleph_{\mathcal{V}}}{|\mathcal{V}|}}\bigg) .$$

We complete the proof of Proposition 2. $\square$

Proof of Proposition 3

Notice that $\aleph_{\mathcal{V}}(s)/\aleph_{\mathcal{V}} \to 0$ for each $1 \le s \le k-2$. By the definition of $\varphi_\ell(\cdot)$, we have $\mathring{\varphi}_\ell(Y_{i_\ell,i'_\ell}) = (-1)^{1-\tau_\ell}\mathring{Y}_{i_\ell,i'_\ell}$ with $\mathring{Y}_{i_\ell,i'_\ell} = Y_{i_\ell,i'_\ell} - \mathbb{E}(Y_{i_\ell,i'_\ell})$. Then we have

$$
\begin{aligned}
\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}} &= \sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}} \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^k \mathring{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\}\big]^{1-\xi_\ell} + o_p\bigg(\sqrt{\frac{\aleph_{\mathcal{V}}}{|\mathcal{V}|}}\bigg)\\
&= \sum_{j=1}^k \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\mathring{\varphi}_j\big(Y_{i_j,i'_j}\big)\prod_{\ell\neq j}\mathbb{E}\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\}\bigg] + o_p\bigg(\sqrt{\frac{\aleph_{\mathcal{V}}}{|\mathcal{V}|}}\bigg) \qquad \text{(S.5)}\\
&= \sum_{j=1}^k \frac{(-1)^{1-\tau_j}}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\mathring{Y}_{i_j,i'_j}\prod_{\ell\neq j}\mathbb{E}\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\}\bigg] + o_p\bigg(\sqrt{\frac{\aleph_{\mathcal{V}}}{|\mathcal{V}|}}\bigg).
\end{aligned}
$$

Notice that $\aleph_{\mathcal{V}}/|\mathcal{V}| = O_p(N^{-1})$ and $\sqrt{N}(\tilde{C}_{\mathcal{V}} - C_{\mathcal{V}}) = (1-\alpha-\beta)^{-k}\sqrt{N}(\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}})$. Then we complete the proof of Proposition 3. $\square$

Proof of Theorem 4

Let

$$
\theta = \mathbb{E}\bigg\{\bigg(\sum_{j=1}^k \frac{(-1)^{1-\tau_j}}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\mathring{Y}_{i_j,i'_j}\prod_{\ell\neq j}\mathbb{E}\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\}\bigg]\bigg)^2\bigg\}.
$$

Based on the Berry-Essen Theorem, we have

$$
\sup_{z\in\mathbb{R}}\bigg|\mathbb{P}\big\{\sqrt{N}(\tilde{C}_{\mathcal{V}} - C_{\mathcal{V}}) \le z\big\} - \Phi\bigg\{\frac{(1-\alpha-\beta)^k z}{\sqrt{N\theta}}\bigg\}\bigg| \to 0\,, \qquad \text{(S.6)}
$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of standard normal distribution. It holds that

$$
\theta = \sum_{j_1,j_2=1}^k \frac{(-1)^{2-\tau_{j_1}-\tau_{j_2}}}{|\mathcal{V}|^2}\sum_{\mathbf{v},\tilde{\mathbf{v}}\in\mathcal{V}}\bigg[\mathbb{E}\big(\mathring{Y}_{i_{j_1},i'_{j_1}}\mathring{Y}_{\tilde{i}_{j_2},\tilde{i}'_{j_2}}\big)\prod_{\ell\neq j_1}\mathbb{E}\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\}\prod_{\ell\neq j_2}\mathbb{E}\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\}\bigg]. \qquad \text{(S.7)}
$$

Notice that $\mathbb{E}(\mathring{Y}_{i_{j_1},i'_{j_1}}\mathring{Y}_{\tilde{i}_{j_2},\tilde{i}'_{j_2}}) = \{A_{i_{j_1},i'_{j_1}}(1-\alpha-\beta)+\alpha\}\{1-\alpha-A_{i_{j_1},i'_{j_1}}(1-\alpha-\beta)\} = \mathrm{Var}(Y_{i_{j_1},i'_{j_1}})$ if $\{i_{j_1},i'_{j_1}\} = \{\tilde{i}_{j_2},\tilde{i}'_{j_2}\}$, and $\mathbb{E}\big(\mathring{Y}_{i_{j_1},i'_{j_1}}\mathring{Y}_{\tilde{i}_{j_2},\tilde{i}'_{j_2}}\big) = 0$ if $\{i_{j_1},i'_{j_1}\} \neq \{\tilde{i}_{j_2},\tilde{i}'_{j_2}\}$. For any $j_1,j_2 = 1,\ldots,k$ and $\mathbf{v} = (i_1,i'_1,\ldots,i_k,i'_k) \in \mathcal{V}$, define $\mathcal{V}_{j_1,j_2}(\mathbf{v}) = \{\tilde{\mathbf{v}} = (\tilde{i}_1,\tilde{i}'_1,\ldots,\tilde{i}_k,\tilde{i}'_k) \in \mathcal{V} : \{\tilde{i}_{j_2},\tilde{i}'_{j_2}\} = \{i_{j_1},i'_{j_1}\}\}$. Then

$$\theta = \sum_{j_1,j_2=1}^{k} \frac{(-1)^{2-\tau_{j_1}-\tau_{j_2}}}{|\mathcal{V}|^2} \sum_{\mathbf{v}\in\mathcal{V}} \bigg[\mathrm{Var}\big(Y_{i_{j_1},i'_{j_1}}\big) \prod_{\ell\neq j_1} \mathbb{E}\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})} \prod_{\ell\neq j_2} \mathbb{E}\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\}\bigg].$$

Define

$$Z = \frac{\sqrt{N}}{(1-\alpha-\beta)^k} \sum_{j=1}^{k} \frac{(-1)^{1-\tau_j}}{|\mathcal{V}|} \sum_{\mathbf{v}\in\mathcal{V}} \bigg\{\mathring{Y}^{\dagger}_{i_j,i'_j} \prod_{\ell\neq j} \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\bigg\}.$$

Given $\mathbf{Y} = (Y_{i,j})_{p\times p}$, we have $Z \to_d \mathcal{N}(0,\hat{\sigma}^2_{\mathcal{V}})$ with

$$\begin{aligned}\hat{\sigma}^2_{\mathcal{V}} &= \frac{1}{(1-\alpha-\beta)^{2k}} \lim_{p\to\infty} N\mathbb{E}^*\Bigg(\bigg[\sum_{j=1}^{k} \frac{(-1)^{1-\tau_j}}{|\mathcal{V}|} \sum_{\mathbf{v}\in\mathcal{V}} \bigg\{\mathring{Y}^{\dagger}_{i_j,i'_j} \prod_{\ell\neq j} \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\bigg\}\bigg]^2\Bigg) \\ &=: \frac{1}{(1-\alpha-\beta)^{2k}} \lim_{p\to\infty} N\theta^*,\end{aligned}$$

where $\mathbb{E}^*(\cdot)$ denotes the conditional expectation given $\mathbf{Y}$. Based on the Berry-Essen Theorem, we have

$$\sup_{z\in\mathbb{R}} \bigg|\mathbb{P}\big(Z \leq z \,|\, \mathbf{Y}\big) - \Phi\bigg\{\frac{(1-\alpha-\beta)^k z}{\sqrt{N\theta^*}}\bigg\}\bigg| \to 0. \tag{S.8}$$

Same as (S.7), we have

$$\begin{aligned}\theta^* &= \sum_{j_1,j_2=1}^{k} \frac{(-1)^{2-\tau_{j_1}-\tau_{j_2}}}{|\mathcal{V}|^2} \sum_{\mathbf{v},\tilde{\mathbf{v}}\in\mathcal{V}} \bigg[\mathbb{E}^*\big(\mathring{Y}^{\dagger}_{i_{j_1},i'_{j_1}}\mathring{Y}^{\dagger}_{\tilde{i}_{j_2},\tilde{i}'_{j_2}}\big) \prod_{\ell\neq j_1} \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) \prod_{\ell\neq j_2} \varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\bigg] \\ &= \sum_{j_1,j_2=1}^{k} \frac{(-1)^{2-\tau_{j_1}-\tau_{j_2}}}{|\mathcal{V}|^2} \sum_{\mathbf{v}\in\mathcal{V}} \bigg[\mathrm{Var}^*\big(Y^{\dagger}_{i_{j_1},i'_{j_1}}\big) \prod_{\ell\neq j_1} \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})} \prod_{\ell\neq j_2} \varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\bigg],\end{aligned}$$

where $\mathrm{Var}^*(\cdot)$ denotes the conditional variance given $\mathbf{Y}$. It follows from (S.6) and (S.8) that

$$
\begin{aligned}
&\sup_{z\in\mathbb{R}}\big|\mathbb{P}\big\{\sqrt{N}(\tilde{C}_{\mathcal{V}}-C_{\mathcal{V}})\le z\big\}-\mathbb{P}\big(Z\le z\,|\,\mathbf{Y}\big)\big| \\
&\qquad\le \sup_{z\in\mathbb{R}}\bigg|\Phi\bigg\{\frac{(1-\alpha-\beta)^k z}{\sqrt{N\theta}}\bigg\}-\Phi\bigg\{\frac{(1-\alpha-\beta)^k z}{\sqrt{N\theta^*}}\bigg\}\bigg|+o(1)\,.
\end{aligned}
$$

In the sequel, we show $|\theta^*-\theta|=o_p(N^{-1})$. To do this, we only need to show

$$
\begin{aligned}
\Delta_{j_1,j_2} :=&\ \frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\mathrm{Var}^*\big(Y^{\dagger}_{i_{j_1},i'_{j_1}}\big)\prod_{\ell\neq j_1}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\bigg] \\
&-\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\mathrm{Var}\big(Y_{i_{j_1},i'_{j_1}}\big)\prod_{\ell\neq j_1}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\bigg] \\
=&\ o_p(N^{-1})
\end{aligned}
$$

for any $j_1,j_2=1,\ldots,k$. Notice that $\mathrm{Var}^*(Y^{\dagger}_{i_j,i'_j})=Y_{i_j,i'_j}(\beta-\alpha)+\alpha(1-\beta)$ and $\mathbb{E}\{\mathrm{Var}^*(Y^{\dagger}_{i_j,i'_j})\}=\mathrm{Var}(Y_{i_j,i'_j})$. Given j_1, define $\tilde{\varphi}_\ell(Y_{i_\ell,i'_\ell})=\varphi_\ell(Y_{i_\ell,i'_\ell})$ for any $\ell\neq j_1$, and $\tilde{\varphi}_{j_1}(Y_{i_{j_1},i'_{j_1}})=Y_{i_{j_1},i'_{j_1}}(\beta-\alpha)+\alpha(1-\beta)$. Then

$$
\begin{aligned}
\Delta_{j_1,j_2} =&\ \frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\prod_{\ell=1}^k\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)-\prod_{\ell=1}^k\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg]\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\} \\
&+\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\prod_{\ell=1}^k\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg]\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\bigg[\prod_{\ell\neq j_2}\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)-\prod_{\ell\neq j_2}\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\bigg] \\
&+\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg[\prod_{\ell=1}^k\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)-\prod_{\ell=1}^k\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg] \\
&\qquad\times\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\bigg[\prod_{\ell\neq j_2}\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)-\prod_{\ell\neq j_2}\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\bigg] \\
:=&\ \Delta_{j_1,j_2}(1)+\Delta_{j_1,j_2}(2)+\Delta_{j_1,j_2}(3)\,.
\end{aligned}
$$

We will show $|\Delta_{j_1,j_2}(1)|=o_p(N^{-1})$, $|\Delta_{j_1,j_2}(2)|=o_p(N^{-1})$ and $|\Delta_{j_1,j_2}(3)|=o_p(N^{-1})$.

For $\Delta_{j_1,j_2}(1)$, it holds that

$$\Delta_{j_1,j_2}(1) = \sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k} \frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^{k}\mathring{\tilde{\varphi}}_\ell(Y_{i_\ell,i'_\ell})^{\xi_\ell}\big[\mathbb{E}\big\{\tilde{\varphi}_\ell(Y_{i_\ell,i'_\ell})\big\}\big]^{1-\xi_\ell}B_1(\mathbf{v})$$

where $\mathring{\tilde{\varphi}}_\ell(Y_{i_\ell,i'_\ell}) = \tilde{\varphi}_\ell(Y_{i_\ell,i'_\ell}) - \mathbb{E}\{\tilde{\varphi}_\ell(Y_{i_\ell,i'_\ell})\}$ and

$$B_1(\mathbf{v}) = \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\,.$$

Same as (S.2) and (S.4), we have

$$\mathbb{E}\big\{|\Delta_{j_1,j_2}(1)|^2\big\} \le \frac{C\aleph_{\mathcal{V}}^3}{|\mathcal{V}|^3} = O(N^{-3})\,,$$

which implies $|\Delta_{j_1,j_2}(1)| = O_p(N^{-3/2}) = o_p(N^{-1})$. Notice that if $\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})$, then $\mathbf{v}\in\mathcal{V}_{j_2,j_1}(\tilde{\mathbf{v}})$. We can reformulate $\Delta_{j_1,j_2}(2)$ as

$$\Delta_{j_1,j_2}(2) = \frac{1}{|\mathcal{V}|^2}\sum_{\tilde{\mathbf{v}}\in\mathcal{V}}\bigg[\prod_{\ell\neq j_2}\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big) - \prod_{\ell\neq j_2}\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\bigg]\sum_{\mathbf{v}\in\mathcal{V}_{j_2,j_1}(\tilde{\mathbf{v}})}\bigg[\prod_{\ell=1}^{k}\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg]\,.$$

Following the same arguments to bound $\mathbb{E}\{|\Delta_{j_1,j_2}(1)|^2\}$, we have $|\Delta_{j_1,j_2}(2)| = o_p(N^{-1})$. For $\Delta_{j_1,j_2}(3)$, we can reformulate it as

$$\begin{aligned}\Delta_{j_1,j_2}(3) = &\sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k}\;\sum_{\substack{\tilde{\xi}_1+\cdots+\tilde{\xi}_{j_2-1}+\tilde{\xi}_{j_2+1}+\cdots+\tilde{\xi}_k=1\\ \tilde{\xi}_1,\ldots,\tilde{\xi}_{j_2-1},\tilde{\xi}_{j_2+1},\ldots,\tilde{\xi}_k\in\{0,1\}}}^{k-1}\\ &\qquad\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg(\prod_{\ell=1}^{k}\mathring{\tilde{\varphi}}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}\bigg)\\ &\qquad\qquad\times\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\bigg(\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\big]^{1-\tilde{\xi}_\ell}\bigg)\,.\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\mathbb{E}\Big\{\Big|\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\Big(\prod_{\ell=1}^{k}\mathring{\tilde{\varphi}}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}\Big)$$

$$\times\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\Big(\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\big]^{1-\tilde{\xi}_\ell}\Big)\Big|^2\Big\}$$

$$\leq\frac{1}{|\mathcal{V}|^4}\mathbb{E}\Big\{\sum_{\mathbf{v}\in\mathcal{V}}\Big(\prod_{\ell=1}^{k}\mathring{\tilde{\varphi}}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}\Big)^2$$

$$\times\sum_{\mathbf{v}\in\mathcal{V}}\Big(\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\big]^{1-\tilde{\xi}_\ell}\Big)^2\Big\}.$$

Notice that $\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)$ is bounded, then

$$\mathbb{E}\Big\{\Big|\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\Big(\prod_{\ell=1}^{k}\mathring{\tilde{\varphi}}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\big\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\big]^{1-\xi_\ell}\Big)$$

$$\times\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\Big(\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\big]^{1-\tilde{\xi}_\ell}\Big)\Big|^2\Big\}$$

$$\leq\frac{C}{|\mathcal{V}|^3}\sum_{\mathbf{v}\in\mathcal{V}}\mathbb{E}\Big\{\Big(\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\big]^{1-\tilde{\xi}_\ell}\Big)^2\Big\}.$$

Same as (S.3), we have

$$\mathbb{E}\Big\{\Big(\sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\big\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\big\}\big]^{1-\tilde{\xi}_\ell}\Big)^2\Big\}\leq C\aleph_{\mathcal{V}}\aleph_{\mathcal{V}}(k-2)\,,$$

which implies

$$
\begin{aligned}
&\mathbb{E}\bigg\{\bigg|\frac{1}{|\mathcal{V}|^2}\sum_{\mathbf{v}\in\mathcal{V}}\bigg(\prod_{\ell=1}^{k}\mathring{\tilde{\varphi}}_\ell\big(Y_{i_\ell,i'_\ell}\big)^{\xi_\ell}\big[\mathbb{E}\{\tilde{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big)\}\big]^{1-\xi_\ell}\bigg) \\
&\qquad\qquad\qquad \times \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{j_1,j_2}(\mathbf{v})}\bigg(\prod_{\ell\neq j_2}\mathring{\varphi}_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)^{\tilde{\xi}_\ell}\big[\mathbb{E}\{\varphi_\ell\big(Y_{\tilde{i}_\ell,\tilde{i}'_\ell}\big)\}\big]^{1-\tilde{\xi}_\ell}\bigg)\bigg|^2\bigg\} \\
&\quad \le \frac{C\aleph_{\mathcal{V}}\aleph_{\mathcal{V}}(k-2)}{|\mathcal{V}|^2} = o\bigg(\frac{\aleph_{\mathcal{V}}^2}{|\mathcal{V}|^2}\bigg) = o(N^{-2})\,.
\end{aligned}
$$

Then $|\Delta_{j_1,j_2}(3)| = o_p(N^{-1})$. We complete the proof of Theorem 4. $\square$

Proof of Proposition 4

To simplify the notation, we write $\hat{T}_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$, $\tilde{T}_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ and $T_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ as $\hat{T}_{\mathcal{V}}$, $\tilde{T}_{\mathcal{V}}$ and $T_{\mathcal{V}}$, respectively. For given $\tau_1,\ldots,\tau_k\in\{0,1\}$, we define $\hat{\varphi}_\ell(x) = (x-\tilde{\alpha})^{\tau_\ell}(1-\tilde{\beta}-x)^{1-\tau_\ell}$ for $x\in\{0,1\}$. Recall that

$$
\tilde{T}_{\mathcal{V}} = \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^{k}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\,.
$$

As we have shown in Proposition 2 that $|\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}}| = O_p(N^{-1/2})$. To show $|\hat{T}_{\mathcal{V}} - T_{\mathcal{V}}| = O_p(N^{-1/2})$, we only need to prove $|\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}}| = O_p(N^{-1/2})$.

For each $\mathbf{v}\in\mathcal{V}$, we have the following identity

$$
\prod_{\ell=1}^{k}\hat{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big) - \prod_{\ell=1}^{k}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) = \sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k}\prod_{\ell=1}^{k}\big\{\hat{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big) - \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}^{\xi_\ell}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}^{1-\xi_\ell}\,.
$$

Recall that $\hat{\varphi}_\ell(Y_{i_\ell,i'_\ell}) - \varphi_\ell(Y_{i_\ell,i'_\ell}) = (\alpha-\tilde{\alpha})^{\tau_\ell}(\beta-\tilde{\beta})^{1-\tau_\ell}$ and $Y_{i_\ell,i'_\ell}\in\{0,1\}$. Let $r_{\max} =$

$\max\{|\tilde{\alpha}-\alpha|, |\tilde{\beta}-\beta|\}$. Notice that $r_{\max} = O_p(N^{-1/2})$. Then

$$
\begin{aligned}
\left|\prod_{\ell=1}^{k}\hat{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big) - \prod_{\ell=1}^{k}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\right| \leq & \sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k} \prod_{\ell=1}^{k}\big(|\hat{\alpha}-\alpha|^{\tau_\ell}|\hat{\beta}-\beta|^{1-\tau_\ell}\big)^{\xi_\ell} \\
\leq & \sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k} r_{\max}^{\xi_1+\cdots+\xi_k} = \sum_{\ell=1}^{k} C_k^\ell r_{\max}^\ell\,,
\end{aligned}
$$

which implies that $|\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}}| \leq \sum_{\ell=1}^{k} C_k^\ell r_{\max}^\ell = O_p(N^{-1/2})$.

Recall that $\tilde{\alpha}-\alpha = O_p(N^{-1/2})$, $\tilde{\beta}-\beta = O_p(N^{-1/2})$ and $\hat{T}_{\mathcal{V}} - T_{\mathcal{V}} = O_p(N^{-1/2})$. It holds that

$$
\begin{aligned}
\sqrt{N}\big(\hat{C}_{\mathcal{V}} - C_{\mathcal{V}}\big) &= \frac{\sqrt{N}\hat{T}_{\mathcal{V}}}{(1-\tilde{\alpha}-\tilde{\beta})^k} - \frac{\sqrt{N}T_{\mathcal{V}}}{(1-\alpha-\beta)^k} \\
&= \frac{\sqrt{N}(\hat{T}_{\mathcal{V}} - T_{\mathcal{V}})}{(1-\alpha-\beta)^k} + \frac{kT_{\mathcal{V}}\sqrt{N}(\tilde{\alpha}-\alpha)}{(1-\alpha-\beta)^{k+1}} + \frac{kT_{\mathcal{V}}\sqrt{N}(\tilde{\beta}-\beta)}{(1-\alpha-\beta)^{k+1}} + O_p(N^{-1/2}) \\
&= \frac{\sqrt{N}(\hat{T}_{\mathcal{V}} - T_{\mathcal{V}})}{(1-\alpha-\beta)^k} + \frac{kC_{\mathcal{V}}\sqrt{N}(\tilde{\alpha}-\alpha)}{1-\alpha-\beta} + \frac{kC_{\mathcal{V}}\sqrt{N}(\tilde{\beta}-\beta)}{1-\alpha-\beta} + O_p(N^{-1/2})\,.
\end{aligned}
\tag{S.9}
$$

In the sequel, we will specify the leading term of $\sqrt{N}(\hat{T}_{\mathcal{V}} - T_{\mathcal{V}})$. Notice that $\sqrt{N}(\hat{T}_{\mathcal{V}} - T_{\mathcal{V}}) = \sqrt{N}(\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}}) + \sqrt{N}(\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}})$. Recall that

$$
\begin{aligned}
\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}} &= \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^{k}\hat{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big) - \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^{k}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) \\
&= \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\sum_{\substack{\xi_1+\cdots+\xi_k=1\\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k}\prod_{\ell=1}^{k}\big\{\hat{\varphi}_\ell\big(Y_{i_\ell,i'_\ell}\big) - \varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}^{\xi_\ell}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}^{1-\xi_\ell}\,.
\end{aligned}
$$

Since $\hat{\varphi}_\ell(Y_{i_\ell,i'_\ell}) - \varphi_\ell(Y_{i_\ell,i'_\ell}) = (\alpha - \tilde{\alpha})^{\tau_\ell}(\beta - \tilde{\beta})^{1-\tau_\ell}$, we have that

$$\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}} = \sum_{\substack{\xi_1+\cdots+\xi_k=1 \\ \xi_1,\ldots,\xi_k\in\{0,1\}}}^{k} (\alpha - \tilde{\alpha})^{\sum_{\ell=1}^k \tau_\ell \xi_\ell}(\beta - \tilde{\beta})^{\sum_{\ell=1}^k (1-\tau_\ell)\xi_\ell} \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^k \big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}^{1-\xi_\ell}\,.$$

If $\sum_{\ell=1}^k \xi_\ell \geq 2$, then

$$(\alpha - \tilde{\alpha})^{\sum_{\ell=1}^k \tau_\ell \xi_\ell}(\beta - \tilde{\beta})^{\sum_{\ell=1}^k (1-\tau_\ell)\xi_\ell} = O_p(N^{-1})\,.$$

for any $\xi_1,\ldots,\xi_k,\tau_1,\ldots,\tau_k \in \{0,1\}$. Due to $|\varphi_\ell(Y_{i_\ell,i'_\ell})| \leq \max\{1-\alpha,\alpha,1-\beta,\beta\}$, then

$$\begin{aligned}
&\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}} \\
&\quad= \sum_{\substack{\xi_1+\cdots+\xi_k=1 \\ \xi_1,\ldots,\xi_k\in\{0,1\}}} (\alpha - \tilde{\alpha})^{\sum_{\ell=1}^k \tau_\ell \xi_\ell}(\beta - \tilde{\beta})^{\sum_{\ell=1}^k (1-\tau_\ell)\xi_\ell} \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell=1}^k \big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}^{1-\xi_\ell} + O_p(N^{-1}) \\
&\quad= \sum_{j=1}^k (\alpha - \tilde{\alpha})^{\tau_j}(\beta - \tilde{\beta})^{1-\tau_j}\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell\neq j}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) + O_p(N^{-1})\,.
\end{aligned}$$

Similar to (S.4), we have

$$\bigg|\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell\neq j}\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big) - \frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell\neq j}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg| = O_p\bigg(\sqrt{\frac{\aleph_{\mathcal{V}}}{|\mathcal{V}|}}\bigg)$$

for any $j = 1,\ldots,k$. Since $\aleph_{\mathcal{V}}/|\mathcal{V}| \asymp N^{-1}$, it holds that

$$\hat{T}_{\mathcal{V}} - \tilde{T}_{\mathcal{V}} = \sum_{j=1}^k (\alpha - \tilde{\alpha})^{\tau_j}(\beta - \tilde{\beta})^{1-\tau_j}\frac{1}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell\neq j}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\} + O_p(N^{-1})\,.$$

As we have shown in (S.5),

$$\tilde{T}_{\mathcal{V}} - T_{\mathcal{V}} = \sum_{j=1}^{k} \frac{(-1)^{1-\tau_j}}{|\mathcal{V}|} \sum_{\mathbf{v}\in\mathcal{V}} \bigg[\mathring{Y}_{i_j,i'_j} \prod_{\ell\neq j} \mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg] + o_p(N^{-1/2})\,.$$

Thus, it follows from (S.9) that

$$\begin{aligned}\sqrt{N}\big(\hat{C}_{\mathcal{V}} - C_{\mathcal{V}}\big) = &\, \frac{\sqrt{N}}{(1-\alpha-\beta)^k} \sum_{j=1}^{k} \frac{(-1)^{1-\tau_j}}{|\mathcal{V}|} \sum_{\mathbf{v}\in\mathcal{V}} \bigg[\mathring{Y}_{i_j,i'_j} \prod_{\ell\neq j} \mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\}\bigg] \\ &- \frac{1}{(1-\alpha-\beta)^k} \sum_{j=1}^{k} \sqrt{N}(\tilde{\alpha}-\alpha)^{\tau_j}(\tilde{\beta}-\beta)^{1-\tau_j} \frac{1}{|\mathcal{V}|} \sum_{\mathbf{v}\in\mathcal{V}} \prod_{\ell\neq j} \mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i'_\ell}\big)\big\} \\ &+ \frac{kC_{\mathcal{V}}\sqrt{N}(\tilde{\alpha}-\alpha)}{1-\alpha-\beta} + \frac{kC_{\mathcal{V}}\sqrt{N}(\tilde{\beta}-\beta)}{1-\alpha-\beta} + o_p(1)\,.\end{aligned}$$

We complete the proof of Proposition 4. □

Proof of Theorem 5

Recall that $\hat{u}_1 - u_1 = (2N)^{-1}\sum_{i\neq j}\mathring{Y}_{i,j}$, $\hat{u}_2 - u_2 = (4N)^{-1}\sum_{i\neq j}\{\eta_{i,j} - \mathbb{E}(\eta_{i,j})\}$ and $\hat{u}_3 - u_3 = (6N)^{-1}\sum_{i\neq j}\{\xi_{i,j} - \mathbb{E}(\xi_{i,j})\}$ with $\eta_{i,j} = |Y_{i,j,*} - Y_{i,j}|$ and $\xi_{i,j} = I(Y_{i,j,**} - 2Y_{i,j,*} + Y_{i,j} = 1 \text{ or} -2)$. Let $\mathring{\eta}_{i,j} = \eta_{i,j} - \mathbb{E}(\eta_{i,j})$ and $\mathring{\xi}_{i,j} = \xi_{i,j} - \mathbb{E}(\xi_{i,j})$. Define $\kappa_1 = \alpha(1-\alpha)$ and $\kappa_2 = \beta(1-\beta)$. Due to $\{(Y_{i,j}, Y_{i,j,*}, Y_{i,j,**})\}_{i<j}$ are independent, and $Y_{i,j} = Y_{j,i}$, $Y_{i,j,*} = Y_{j,i,*}$ and $Y_{i,j,**} = Y_{j,i,**}$, thus $\mathbb{E}(\mathring{Y}_{s_1,t_1}\mathring{Y}_{s_2,t_2}) = A_{s_1,t_1}\kappa_2 + (1 - A_{s_1,t_1})\kappa_1$ if $\{s_1,t_1\} = \{s_2,t_2\}$, $\mathbb{E}(\mathring{Y}_{s_1,t_1}\mathring{Y}_{s_2,t_2}) = 0$ if $\{s_1,t_1\} \neq \{s_2,t_2\}$, $\mathbb{E}(\mathring{Y}_{s_1,t_1}\mathring{\eta}_{s_2,t_2}) = A_{s_1,t_1}\kappa_2(2\beta - 1) + (1 - A_{s_1,t_1})\kappa_1(1-2\alpha)$ if $\{s_1,t_1\} = \{s_2,t_2\}$, $\mathbb{E}(\mathring{Y}_{s_1,t_1}\mathring{\eta}_{s_2,t_2}) = 0$ if $\{s_1,t_1\} \neq \{s_2,t_2\}$, $\mathbb{E}(\mathring{Y}_{s_1,t_1}\mathring{\xi}_{s_2,t_2}) = A_{s_1,t_1}\kappa_2(\beta^2 - 2\kappa_2) + (1 - A_{s_1,t_1})\kappa_1\{(1-\alpha)^2 - 2\kappa_1\}$ if $\{s_1,t_1\} = \{s_2,t_2\}$ and $\mathbb{E}(\mathring{Y}_{s_1,t_1}\mathring{\xi}_{s_2,t_2}) = 0$ if $\{s_1,t_1\} \neq \{s_2,t_2\}$. Notice that $\sqrt{N}\{\hat{C}_{\mathcal{V}}(\tau_1,\ldots,\tau_k) - C_{\mathcal{V}}(\tau_1,\ldots,\tau_k)\} = S_{\mathcal{V}}(\tau_1,\ldots,\tau_k) + \Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ with $\Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k) = \Delta_{\alpha,\mathcal{V}}(\tau_1,\ldots,\tau_k)\sqrt{N}(\tilde{\alpha}-\alpha) + \Delta_{\beta,\mathcal{V}}(\tau_1,\ldots,\tau_k)\sqrt{N}(\tilde{\beta}-\beta)$. The asymptotic variances of $S_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ has been specified in (22) and the asymptotic variance of $\Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ can be obtained via Theorems 2 and 3. Here we only need to

specify $\mathrm{Cov}\{S_{\mathcal{V}}(\tau_1,\ldots,\tau_k),\Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k)\}$. Due to $\Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ is a linear combination of $\sqrt{N}(\tilde{\alpha}-\alpha)$ and $\sqrt{N}(\tilde{\beta}-\beta)$, and the leading terms of $\tilde{\alpha}-\alpha$ and $\tilde{\beta}-\beta$ are both linear combinations of $\hat{u}_1-u_1$, $\hat{u}_2-u_2$ and $\hat{u}_3-u_3$, then the leading term of $\Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k)$ is also a linear combination of $\hat{u}_1-u_1$, $\hat{u}_2-u_2$ and $\hat{u}_3-u_3$. We first calculate a more general result $\mathrm{Cov}\{S_{\mathcal{V}}(\tau_1,\ldots,\tau_k),x_1\sqrt{N}(\hat{u}_1-u_1)+x_2\sqrt{N}(\hat{u}_2-u_2)+x_3\sqrt{N}(\hat{u}_3-u_3)\}$ for any $(x_1,x_2,x_3)\in\mathbb{R}^3$.

Notice that

$$\begin{aligned}
&\mathrm{Cov}\big\{S_{\mathcal{V}}(\tau_1,\ldots,\tau_k),x_1\sqrt{N}(\hat{u}_1-u_1)+x_2\sqrt{N}(\hat{u}_2-u_2)+x_3\sqrt{N}(\hat{u}_3-u_3)\big\}\\
&=\frac{x_1}{2(1-\alpha-\beta)^k}\sum_{j=1}^k\frac{(-1)^{1-\tau_j}}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\sum_{s\neq t}\mathbb{E}\big(\mathring{Y}_{i_j,i_j'}\mathring{Y}_{s,t}\big)\prod_{\ell\neq j}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i_\ell'}\big)\big\}\\
&\quad+\frac{x_2}{4(1-\alpha-\beta)^k}\sum_{j=1}^k\frac{(-1)^{1-\tau_j}}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\sum_{s\neq t}\mathbb{E}\big(\mathring{Y}_{i_j,i_j'}\mathring{\eta}_{s,t}\big)\prod_{\ell\neq j}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i_\ell'}\big)\big\}\\
&\quad+\frac{x_3}{6(1-\alpha-\beta)^k}\sum_{j=1}^k\frac{(-1)^{1-\tau_j}}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\sum_{s\neq t}\mathbb{E}\big(\mathring{Y}_{i_j,i_j'}\mathring{\xi}_{s,t}\big)\prod_{\ell\neq j}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i_\ell'}\big)\big\}\\
&=\bigg\{\frac{x_1(\kappa_2-\kappa_1)}{1-\alpha-\beta}+\frac{x_2\{\kappa_2(2\beta-1)-\kappa_1(1-2\alpha)\}}{2(1-\alpha-\beta)}\\
&\qquad\qquad+\frac{x_3[\kappa_2(\beta^2-2\kappa_2)-\kappa_1\{(1-\alpha)^2-2\kappa_1\}]}{3(1-\alpha-\beta)}\bigg\}\\
&\qquad\qquad\times\sum_{j=1}^k(-1)^{1-\tau_j}C_{\mathcal{V}}(\tau_1,\ldots,\tau_{j-1},1,\tau_{j+1},\ldots,\tau_k)\\
&\quad+\bigg[\frac{x_1\kappa_1}{(1-\alpha-\beta)^k}+\frac{x_2\kappa_1(1-2\alpha)}{2(1-\alpha-\beta)^k}+\frac{x_3\kappa_1\{(1-\alpha)^2-2\kappa_1\}}{3(1-\alpha-\beta)^k}\bigg]\\
&\qquad\qquad\times\sum_{j=1}^k\frac{(-1)^{1-\tau_j}}{|\mathcal{V}|}\sum_{\mathbf{v}\in\mathcal{V}}\prod_{\ell\neq j}\mathbb{E}\big\{\varphi_\ell\big(Y_{i_\ell,i_\ell'}\big)\big\}\,.
\end{aligned}\tag{S.10}$$

If α is known, we have $\tilde{\alpha}=\alpha$ and $\tilde{\beta}=\hat{\beta}$. Then $\Xi_{\mathcal{V}}(\tau_1,\ldots,\tau_k)=\Delta_{\beta,\mathcal{V}}(\tau_1,\ldots,\tau_k)\sqrt{N}(\hat{\beta}-\beta)$. As we have shown in the proof of Theorem 2, $\hat{\beta}-\beta=g_{\beta,1}(\hat{u}_1-u_1)+g_{\beta,2}(\hat{u}_2-u_2)+o_p(N^{-1/2})$. With selecting $x_1=g_{\beta,1}\Delta_{\beta,\mathcal{V}}(\tau_1,\ldots,\tau_k)$, $x_2=g_{\beta,2}\Delta_{\beta,\mathcal{V}}(\tau_1,\ldots,\tau_k)$ and $x_3=0$

in (S.10), we then have part (i).

If β is known, we have $\tilde{\alpha} = \hat{\alpha}$ and $\tilde{\beta} = \beta$. Then $\Xi_{\mathcal{V}}(\tau_1, \dots, \tau_k) = \Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k)\sqrt{N}(\hat{\alpha} - \alpha)$. As we have shown in the proof of Theorem 2, $\hat{\alpha} - \alpha = g_{\alpha,1}(\hat{u}_1 - u_1) + g_{\alpha,2}(\hat{u}_2 - u_2) + o_p(N^{-1/2})$. With selecting $x_1 = g_{\alpha,1}\Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k)$, $x_2 = g_{\alpha,2}\Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k)$ and $x_3 = 0$ in (S.10), we then have part (ii).

If α and β are unknown, we have $\tilde{\alpha} = \hat{\alpha}$ and $\tilde{\beta} = \hat{\beta}$. Then $\Xi_{\mathcal{V}}(\tau_1, \dots, \tau_k) = \Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k)\sqrt{N}(\hat{\alpha} - \alpha) + \Delta_{\beta,\mathcal{V}}(\tau_1, \dots, \tau_k)\sqrt{N}(\hat{\beta} - \beta)$. As we have shown in the proof of Theorem 3, $\hat{\alpha} - \alpha = g_{\alpha,1}(\hat{u}_1 - u_1) + g_{\alpha,2}(\hat{u}_2 - u_2) + g_{\alpha,3}(\hat{u}_3 - u_3) + o_p(N^{-1/2})$ and $\hat{\beta} - \beta = g_{\beta,1}(\hat{u}_1 - u_1) + g_{\beta,2}(\hat{u}_2 - u_2) + g_{\beta,3}(\hat{u}_3 - u_3) + o_p(N^{-1/2})$. With selecting $x_1 = g_{\alpha,1}\Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k) + g_{\beta,1}\Delta_{\beta,\mathcal{V}}(\tau_1, \dots, \tau_k)$, $x_2 = g_{\alpha,2}\Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k) + g_{\beta,2}\Delta_{\beta,\mathcal{V}}(\tau_1, \dots, \tau_k)$ and $x_3 = g_{\alpha,3}\Delta_{\alpha,\mathcal{V}}(\tau_1, \dots, \tau_k) + g_{\beta,3}\Delta_{\beta,\mathcal{V}}(\tau_1, \dots, \tau_k)$ in (S.10), we then have part (iii). $\square$

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