Appendix 1: Derivation of the matching priors for α in Theorem 2.1

In finding the matching priors for α, we consider the orthogonal parameterization on nuisance parameters as follows. Let

$$\theta_1 = \alpha, \theta_2 = \phi\alpha^{\frac{3}{2}}(2+\alpha)^{-\frac{1}{2}} \text{ and } \theta_3 = \beta\alpha^{\frac{3}{2}}(2+\alpha)^{-\frac{1}{2}}.$$

With this parametrization, the likelihood function of parameters $(\theta_1, \theta_2, \theta_3)$ for the model is given by

$$L(\theta_1, \theta_2, \theta_3) \quad \propto \quad \frac{\theta_1^{-2}(\theta_1+1)(2+\theta_1)\theta_2\theta_3}{\left(1+\theta_3\theta_1^{-\frac{3}{2}}(2+\theta_1)^{\frac{1}{2}}x + \theta_2\theta_1^{-\frac{3}{2}}(2+\theta_1)^{\frac{1}{2}}y\right)^{\alpha+2}}. \tag{1}$$

Based on (1), the Fisher information matrix is given by

$$\mathbf{I}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \frac{4+6\theta_1+3\theta_1^2}{\theta_1^2(1+\theta_1)^2(2+\theta_1)^2} & 0 & 0 \\ 0 & \frac{1+\theta_1}{(3+\theta_1)\theta_2^2} & -\frac{1}{(3+\theta_1)\theta_2\theta_3} \\ 0 & -\frac{1}{(3+\theta_1)\theta_2\theta_3} & \frac{1+\theta_1}{(3+\theta_1)\theta_3^2} \end{pmatrix}. \tag{2}$$

From the above Fisher information matrix $\mathbf{I}$, θ_1 is orthogonal to (θ_2, θ_3) in the sense of Cox and Reid(1987). Following Tibshirani(1989), the class of first order probability matching prior is characterized by

$$\pi_m^{(1)}(\theta_1, \theta_2, \theta_3) \propto (4+6\theta_1+3\theta_1^2)^{\frac{1}{2}}\theta_1^{-1}(1+\theta_1)^{-1}(2+\theta_1)^{-1}d(\theta_2, \theta_3), \tag{3}$$

where $d(\theta_2, \theta_3) > 0$ is an arbitrary function differentiable in its arguments. The class of prior given in (3) can be narrowed down to the second order probability matching priors as given in Mukerjee and Ghosh (1997). A second order probability matching prior is of the form (3), and also d must satisfy an additional differential equation (2.10) of Mukerjee and Ghosh (1997),

$$\frac{1}{6}d(\theta_2, \theta_3)\frac{\partial}{\partial\theta_1}\{I_{11}^{-\frac{3}{2}}L_{1,1,1}\} + \sum_{s=2}^{3}\sum_{u=2}^{3}\frac{\partial}{\partial\theta_u}\{I_{11}^{-\frac{1}{2}}L_{11s}I^{su}d(\theta_2, \theta_3)\} = 0, \tag{4}$$

where the inverse matrix of Fisher information matrix $\mathbf{I}^{-1} = (I^{ij})_{3\times 3}$. Then

$$L_{1,1,1} = E\left[\left(\frac{\partial \log L}{\partial \theta_1}\right)^3\right] = -\frac{2(32+116\theta_1+144\theta_1^2+81\theta_1^3+18\theta_1^4)}{\theta_1^3(1+\theta_1)^3(2+\theta_1)^3(4+\theta_1)}, \tag{5}$$

$$L_{112} = E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_2}\right] = -\frac{3}{\theta_1(2+\theta_1)(3+\theta_1)(4+\theta_1)\theta_2}, \tag{6}$$

$$L_{113} = E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_3}\right] = -\frac{3}{\theta_1(2+\theta_1)(3+\theta_1)(4+\theta_1)\theta_3}. \tag{7}$$

and

$$I^{22} = \frac{(1+\theta_1)(3+\theta_1)\theta_2^2}{\theta_1(2+\theta_1)}, I^{23} = \frac{(3+\theta_1)\theta_2\theta_3}{\theta_1(2+\theta_1)}, I^{33} = \frac{(1+\theta_1)(3+\theta_1)\theta_3^2}{\theta_1(2+\theta_1)}. \tag{8}$$

Thus from (5), $\cdots$, (8) and Fisher information matrix (2), the differential equation (4) simplifies to

$$\frac{\theta_1(4+\theta_1)(4+6\theta_1+3\theta_1^2)^{\frac{1}{2}}}{9(1+\theta_1)}\frac{\partial}{\partial\theta_1}\left\{\frac{32+116\theta_1+144\theta_1^2+81\theta_1^3+18\theta_1^4}{(4+\theta_1)(4+6\theta_1+3\theta_1^2)^{\frac{3}{2}}}\right\}$$
$$+d(\theta_2,\theta_3)^{-1}\frac{\partial}{\partial\theta_2}\{\theta_2 d(\theta_2,\theta_3)\} + d(\theta_2,\theta_3)^{-1}\frac{\partial}{\partial\theta_3}\{\theta_3 d(\theta_2,\theta_3)\} = 0. \tag{9}$$

Note that the first left in the differential equation (9) is the function of θ_1 only, and the second and third right part in the differential equation (9) are the functions of θ_2 and θ_3 only. Thus, there can be no solution to the differential equation (9) unless the d is the functions of all θ_1, θ_2 and θ_3. Therefore the second order matching prior does not exist.

We may note that the first order matching prior (3) in the original parametrization (α, β, ϕ) is given by

$$\pi_m(\alpha,\beta,\phi) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}\alpha^2}{(1+\alpha)(2+\alpha)^2} d\left(\phi\alpha^{\frac{3}{2}}(2+\alpha)^{-\frac{1}{2}}, \beta\alpha^{\frac{3}{2}}(2+\alpha)^{-\frac{1}{2}}\right). \tag{10}$$

Appendix 2: Derivation of the matching priors for β in Theorem 2.1

In this appendix, we want to derive the matching priors for β. Then we consider the following orthogonal reparametrization.

$$\theta_1 = \beta, \theta_2 = (2+\alpha)e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}}\phi \text{ and } \theta_3 = \frac{\alpha(1+\alpha)}{2+\alpha}e^{-\frac{2}{\alpha(2+\alpha)}}\beta.$$

Then the Jacobian matrix of this transformation is

$$\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(\beta,\alpha,\phi)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(4+12\alpha+20\alpha^2+17\alpha^3+7\alpha^4+\alpha^5)}{\alpha^2(1+\alpha)^2(2+\alpha)} e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}}\phi & (2+\alpha)e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}} \\ \frac{\alpha(1+\alpha)}{2+\alpha}e^{-\frac{2}{\alpha(2+\alpha)}} & \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)}{\alpha(2+\alpha)^3}e^{-\frac{2}{\alpha(2+\alpha)}}\beta & 0 \end{pmatrix} \quad (11)$$

And its determinant is

$$\frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)}{\alpha(2+\alpha)^2}e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}}\beta.$$

Then the inverse of the expected Fisher information matrix can be written as

$$\begin{aligned} I^{-1}(\theta_1,\theta_2,\theta_3) &= \left(\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(\beta,\alpha,\phi)}\right) I^{-1}(\beta,\alpha,\phi) \left(\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(\beta,\alpha,\phi)}\right)^t \\ &= \begin{pmatrix} \frac{(1+\alpha)(3+\alpha)w_1(\alpha)\beta^2}{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)} & 0 & 0 \\ 0 & \frac{w_2(\alpha)\phi^2}{\alpha^2(1+\alpha)^2}e^{-\frac{2(2+6\alpha+3\alpha^2)}{\alpha(2+3\alpha+\alpha^2)}} & \frac{w_1(\alpha)\beta\phi}{\alpha(2+\alpha)^2}e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}} \\ 0 & \frac{w_1(\alpha)\beta\phi}{\alpha(2+\alpha)^2}e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}} & \frac{w_1(\alpha)(1+\alpha)^2\beta^2}{(2+\alpha)^4}e^{-\frac{4}{\alpha(2+\alpha)}} \end{pmatrix} \quad (12) \end{aligned}$$

where

$$\begin{aligned} w_1(\alpha) &= 4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4, \\ w_2(\alpha) &= 4+12\alpha+26\alpha^2+34\alpha^3+24\alpha^4+8\alpha^5+\alpha^6. \end{aligned}$$

Therefore from (12), the Fisher information matrix is given by

$$I(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)w_1(\alpha)\beta^2} & 0 & 0 \\ 0 & \frac{1+\alpha}{(2+\alpha)^2(3+\alpha)\phi^2}e^{\frac{2(2+6\alpha+3\alpha^2)}{\alpha(2+3\alpha+\alpha^2)}} & -\frac{1}{\alpha(3+4\alpha+\alpha^2)\phi\beta}e^{\frac{2+3\alpha}{\alpha(1+\alpha)}} \\ 0 & -\frac{1}{\alpha(3+4\alpha+\alpha^2)\phi\beta}e^{\frac{2+3\alpha}{\alpha(1+\alpha)}} & \frac{w_2(\alpha)(2+\alpha)^2}{w_3(\alpha)\alpha^2(1+\alpha)^3\beta^2}e^{\frac{4}{\alpha(2+\alpha)}} \end{pmatrix}, \quad (13)$$

where

$$w_3(\alpha) = 12+40\alpha+54\alpha^2+32\alpha^3+9\alpha^4+\alpha^5.$$

Since θ_1 is orthogonal to θ_2 and θ_3, the class of first order probability matching prior is given by

$$\pi_m(\theta_1,\theta_2,\theta_3) \propto \left[\frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)w_1(\alpha)}\right]^{\frac{1}{2}} \beta^{-1} d(\theta_2,\theta_3), \quad (14)$$

where $d(\theta_2, \theta_3) > 0$ is an arbitrary function differentiable in its argument.

In the original parametrization (β, α, ϕ), we may also note that the matching prior (14) is given by

$$\begin{aligned}\pi_m(\beta, \alpha, \phi) \quad &\propto \quad \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}(2+\alpha)^{\frac{3}{2}}(3+\alpha)^{\frac{1}{2}}} \\ &\times \quad e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}} d\left((2+\alpha)e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}}\phi, \frac{\alpha(1+\alpha)}{2+\alpha}e^{-\frac{2}{\alpha(2+\alpha)}}\beta\right). \end{aligned} \tag{15}$$

Appendix 3: Derivation of the matching priors for ϕ in Theorem 2.1

We develop the matching priors for ϕ in this appendix. Then we consider the following orthogonal reparametrization.

$$\theta_1 = \phi, \theta_2 = \frac{\alpha(1+\alpha)}{2+\alpha}e^{-\frac{2}{\alpha(2+\alpha)}}\phi \text{ and } \theta_3 = (2+\alpha)e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}}\beta.$$

Then the Jacobian matrix of this transformation is

$$\begin{aligned}&\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\phi, \alpha, \beta)} \\ = &\begin{pmatrix} 1 & 0 & 0 \\ \frac{\alpha(1+\alpha)}{2+\alpha}e^{-\frac{2}{\alpha(2+\alpha)}} & \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)}{\alpha(2+\alpha)^3}e^{-\frac{2}{\alpha(2+\alpha)}}\phi & 0 \\ 0 & \frac{(4+12\alpha+20\alpha^2+17\alpha^3+7\alpha^4+\alpha^5)}{\alpha^2(1+\alpha)^2(2+\alpha)}e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}}\beta & (2+\alpha)e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}} \end{pmatrix}\end{aligned} \tag{16}$$

And its determinant is

$$\frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)}{\alpha(2+\alpha)^2}e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}}\phi.$$

Then the inverse of the expected Fisher information matrix can be written as

$$\begin{aligned} I^{-1}(\theta_1, \theta_2, \theta_3) \quad &= \quad \left(\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\beta, \alpha, \phi)}\right) I^{-1}(\beta, \alpha, \phi) \left(\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\beta, \alpha, \phi)}\right)^t \\ &= \quad \begin{pmatrix} \frac{(1+\alpha)(3+\alpha)w_1(\alpha)\phi^2}{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)} & 0 & 0 \\ 0 & \frac{w_1(\alpha)(1+\alpha)^2\phi^2}{(2+\alpha)^4}e^{-\frac{4}{\alpha(2+\alpha)}} & \frac{w_1(\alpha)\beta\phi}{\alpha(2+\alpha)^2}e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}} \\ 0 & \frac{w_1(\alpha)\beta\phi}{\alpha(2+\alpha)^2}e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}} & \frac{w_2(\alpha)\beta^2}{\alpha^2(1+\alpha)^2}e^{-\frac{2(2+6\alpha+3\alpha^2)}{\alpha(2+3\alpha+\alpha^2)}} \end{pmatrix} \end{aligned} \tag{17}$$

where

$$
\begin{aligned}
w_1(\alpha) &= 4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4, \\
w_2(\alpha) &= 4 + 12\alpha + 26\alpha^2 + 34\alpha^3 + 24\alpha^4 + 8\alpha^5 + \alpha^6.
\end{aligned}
$$

Therefore from (17), the Fisher information matrix is given by

$$
I(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)w_1(\alpha)\phi^2} & 0 & 0 \\ 0 & \frac{w_2(\alpha)(2+\alpha)^2}{w_3(\alpha)\alpha^2(1+\alpha)^3\phi^2} e^{\frac{4}{\alpha(2+\alpha)}} & -\frac{1}{\alpha(3+4\alpha+\alpha^2)\phi\beta} e^{\frac{2+3\alpha}{\alpha(1+\alpha)}} \\ 0 & -\frac{1}{\alpha(3+4\alpha+\alpha^2)\phi\beta} e^{\frac{2+3\alpha}{\alpha(1+\alpha)}} & \frac{1+\alpha}{(2+\alpha)^2(3+\alpha)\beta^2} e^{\frac{2(2+6\alpha+3\alpha^2)}{\alpha(2+3\alpha+\alpha^2)}} \end{pmatrix}, \quad (18)
$$

where

$$
w_3(\alpha) = 12 + 40\alpha + 54\alpha^2 + 32\alpha^3 + 9\alpha^4 + \alpha^5.
$$

Thus θ_1 is orthogonal to θ_2 and θ_3. Then the class of first order probability matching prior is characterized by

$$
\pi_m(\theta_1, \theta_2, \theta_3) \propto \left[\frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)w_1(\alpha)} \right]^{\frac{1}{2}} \phi^{-1} d(\theta_2, \theta_3), \quad (19)
$$

where $d(\theta_2, \theta_3) > 0$ is an arbitrary function differentiable in its argument.

We may note that the matching prior (19) in the original parametrization (ϕ, α, β) is given by

$$
\begin{aligned}
\pi_m(\phi, \alpha, \beta) &\propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}(2+\alpha)^{\frac{3}{2}}(3+\alpha)^{\frac{1}{2}}} \\
&\times \; e^{-\frac{2+3\alpha}{\alpha(1+\alpha)}} d\left(\frac{\alpha(1+\alpha)}{2+\alpha} e^{-\frac{2}{\alpha(2+\alpha)}} \phi, (2+\alpha) e^{-\frac{2+6\alpha+3\alpha^2}{\alpha(2+3\alpha+\alpha^2)}} \beta \right). \quad (20)
\end{aligned}
$$

Appendix 4: Proof of Theorem 3.1

We compute the Fisher information matrix and the inverse matrix of the Fisher information matrix required to derive reference priors. From the likelihood (**??**), the Fisher information matrix is given by

$$
I(\alpha, \beta, \phi) = \begin{pmatrix} \frac{1+2\alpha+2\alpha^2}{\alpha^2(1+\alpha)^2} & \frac{1}{(2+\alpha)\beta} & \frac{1}{(2+\alpha)\phi} \\ \frac{1}{(2+\alpha)\beta} & \frac{1+\alpha}{(3+\alpha)\beta^2} & -\frac{1}{(3+\alpha)\beta\phi} \\ \frac{1}{(2+\alpha)\phi} & -\frac{1}{(3+\alpha)\beta\phi} & \frac{1+\alpha}{(3+\alpha)\phi^2} \end{pmatrix}. \quad (21)
$$

and so its inverse matrix is given by

$$I^{-1}(\alpha,\beta,\phi)=\begin{pmatrix} \frac{\alpha^2(1+\alpha)^2(2+\alpha)^2}{4+6\alpha+3\alpha^2} & -\frac{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)\beta}{4+6\alpha+3\alpha^2} & -\frac{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)\phi}{4+6\alpha+3\alpha^2} \\ -\frac{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)\beta}{4+6\alpha+3\alpha^2} & \frac{(1+\alpha)(3+\alpha)w_1(\alpha)\beta^2}{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)} & \frac{(3+\alpha)w_4(\alpha)\beta\phi}{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)} \\ -\frac{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)\phi}{4+6\alpha+3\alpha^2} & \frac{(3+\alpha)w_4(\alpha)\beta\phi}{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)} & \frac{(1+\alpha)(3+\alpha)w_1(\alpha)\phi^2}{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)} \end{pmatrix}, \quad (22)$$

where $w_1(\alpha)=4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4$ and $w_4(\alpha)=4+12\alpha+20\alpha^2+17\alpha^3+7\alpha^4+\alpha^5$.

In this appdendix, we consider the grouping orders $\{\alpha,(\beta,\phi)\}$, $\{(\beta,\phi),\alpha\}$, $\{(\alpha,\beta),\phi\}$, $\{(\alpha,\phi),\beta\}$, $\{\alpha,\beta,\phi\}$ and $\{\alpha,\phi,\beta\}$ in development of the reference priors for α. First, we derive the two group reference prior for the parameter grouping $\{\alpha,(\beta,\phi)\}$.

The compact subsets were taken to be Cartesian products of sets of the form

$$\alpha\in[a_1,b_1],\beta\in[a_2,b_2],\phi\in[a_3,b_3]. \quad (23)$$

In the limit $a_i, i=1,2,3$ will tend to 0, and $b_i, i=1,2,3$ will tend to ∞. From the Fisher information (21), we obtain determinant of the Fisher information (21) as follows.

$$|I(\alpha,\beta,\phi)| \;=\; \frac{4+6\alpha+3\alpha^2}{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)^2\beta^2\phi^2}. \quad (24)$$

Also determinant of the Fisher information $I_2(\beta,\phi)$ is given by

$$|I_2(\beta,\phi)| \;=\; \frac{\alpha(2+\alpha)}{(3+\alpha)^2\beta^2\phi^2}. \quad (25)$$

Thus we have

$$h_1=\frac{|I(\alpha,\beta,\phi)|}{|I_2(\beta,\phi)|}=\frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}, h_2=|I_2(\beta,\phi)|=\frac{\alpha(2+\alpha)}{(3+\alpha)^2\beta^2\phi^2}.$$

Here, and below, a subscripted Q denotes a function that is constant and does not depend on any parameters but any Q may depend on the ranges of the parameters.

Step 1. Note that

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2} h_2^{1/2}d\beta d\phi=\int_{a_3}^{b_3}\int_{a_2}^{b_2}\left[\frac{\alpha(2+\alpha)}{(3+\alpha)^2\beta^2\phi^2}\right]^{\frac{1}{2}}d\beta d\phi=\left[\frac{\alpha(2+\alpha)}{(3+\alpha)^2}\right]^{\frac{1}{2}}Q_1.$$

It follows that

$$\pi_2^l(\beta,\phi|\alpha)=Q_1^{-1}\beta^{-1}\phi^{-1}.$$

Step 2. We have

$$\begin{aligned} E^l\{\log h_1|\beta,\phi\} &= \int_{a_3}^{b_3}\int_{a_2}^{b_2} \log\left(\frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right)\pi_2^l(\beta,\phi|\alpha)d\beta d\phi \\ &= \log\left(\frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right). \end{aligned}$$

It follows that

$$\pi_1^l(\alpha) \propto \exp[E^l\{\log h_1|\beta,\phi\}/2] = \left(\frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right)^{\frac{1}{2}}.$$

Therefore the two group reference prior is

$$\pi_{r1}(\alpha,\beta,\phi) = \lim_{l\to\infty}\frac{\pi_2^l(\beta,\phi|\alpha)\pi_1^l(\alpha)}{\pi_2^l(\beta_0,\phi_0|\alpha_0)\pi_1^l(\alpha_0)} \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}, \tag{26}$$

where $\alpha_0, \beta_0, \phi_0$ are an inner point of the interval $(0,\infty)$.

Second, we derive the two group reference prior for the parameter grouping $\{(\beta,\phi),\alpha\}$. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (21).

$$h_1 = \frac{\alpha(4+6\alpha+3\alpha^2)}{(2+\alpha)(3+\alpha)^2(1+2\alpha+2\alpha^2)\beta^2\phi^2} \text{ and } h_2 = \frac{(1+2\alpha+2\alpha^2)}{\alpha^2(1+\alpha)^2}.$$

Then by the derivation procedure of the first reference prior for α, we can show that the two group reference prior is

$$\pi_{r2}(\alpha,\phi,\beta) \propto \frac{(1+2\alpha+2\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)}\beta^{-1}\phi^{-1}. \tag{27}$$

Third, we derive the two group reference prior for the parameter grouping $\{(\alpha,\beta),\phi\}$. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (21).

$$h_1 = \frac{4+6\alpha+3\alpha^2}{\alpha(1+\alpha)^3(2+\alpha)(3+\alpha)\beta^2} \text{ and } h_2 = \frac{1+\alpha}{(3+\alpha)\phi^2}.$$

Then by the derivation procedure of the first reference prior for α, we can show that the two group reference prior is

$$\pi_{r3}(\alpha,\phi,\beta) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)^{\frac{3}{2}}(2+\alpha)^{\frac{1}{2}}(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1}. \tag{28}$$

Fourth, we derive the two group reference prior for the parameter grouping $\{(\alpha, \phi), \beta\}$. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (21).

$$h_1 = \frac{4 + 6\alpha + 3\alpha^2}{\alpha(1+\alpha)^3(2+\alpha)(3+\alpha)\phi^2} \text{ and } h_2 = \frac{1+\alpha}{(3+\alpha)\beta^2}.$$

Then by the derivation procedure of the first reference prior for α, we can show that the two group reference prior is

$$\pi_{r4}(\alpha, \phi, \beta) \propto \frac{(4 + 6\alpha + 3\alpha^2)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)^{\frac{3}{2}}(2+\alpha)^{\frac{1}{2}}(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1}. \tag{29}$$

Fifth, we derive the one-at-a-time reference prior for the parameter grouping $\{\alpha, \beta, \phi\}$. For the derivation of the reference prior, we obtain the following quantities from the inverse matrix of the Fisher information (22).

$$h_1 = \frac{4 + 6\alpha + 3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}, h_2 = \frac{\alpha(2+\alpha)}{(1+\alpha)(3+\alpha)\beta^2} \text{ and } h_3 = \frac{1+\alpha}{(3+\alpha)\phi^2}.$$

Step 1. Note that

$$\int_{a_3}^{b_3} h_3^{1/2} d\phi = \int_{a_3}^{b_3} \left[\frac{1+\alpha}{(3+\alpha)\phi^2}\right]^{1/2} d\phi = \left[\frac{1+\alpha}{3+\alpha}\right]^{1/2} Q_1.$$

It follows that

$$\pi_3^l(\phi|\alpha, \beta) = \frac{h_3^{1/2}}{\int_{a_3}^{b_3} h_3^{1/2} d\sigma} = Q_1^{-1}\phi^{-1}.$$

Step 2. Now we have

$$E^l\{\log h_2|\alpha, \beta\} = \int_{a_3}^{b_3} Q_1^{-1}\phi^{-1} \log\left[\frac{\alpha(2+\alpha)}{(1+\alpha)(3+\alpha)\beta^2}\right] d\phi = \log\left[\frac{\alpha(2+\alpha)}{(1+\alpha)(3+\alpha)\beta^2}\right].$$

It follows that

$$\int_{a_2}^{b_2} \exp[E^l\{\log h_2|\alpha, \beta\}/2] d\beta = \left[\frac{\alpha(2+\alpha)}{(1+\alpha)(3+\alpha)}\right]^{1/2} Q_2.$$

Hence

$$\pi_2^l(\beta, \phi|\alpha) = \frac{\pi_3^l(\phi|\alpha, \beta)\exp[E^l\{\log h_2|\alpha, \beta\}/2]}{\int_{a_2}^{b_2} \exp[E^l\{\log h_2|\alpha, \beta\}/2] d\beta} = Q_1^{-1}Q_2^{-1}\beta^{-1}\phi^{-1}.$$

Step 3. In the final step,

$$\begin{aligned} E^l\{\log h_1|\alpha\} &= \int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_1^{-1}Q_2^{-1}\beta^{-1}\phi^{-1}\log\left(\frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right)d\phi d\beta \\ &= \log\left(\frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right). \end{aligned}$$

It follows that

$$\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\alpha\}/2]d\alpha = Q_3.$$

Hence

$$\begin{aligned} \pi_1^l(\alpha,\beta,\phi) &= \frac{\pi_2^l(\beta,\phi|\alpha)\exp[E^l\{\log h_1|\alpha\}/2]}{\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\alpha\}/2]d\alpha} \\ &= Q_1^{-1}Q_2^{-1}Q_3^{-1}(4+6\alpha+3\alpha^2)^{\frac{1}{2}}\alpha^{-1}(1+\alpha)^{-1}(2+\alpha)^{-1}\beta^{-1}\phi^{-1}. \end{aligned}$$

Thus the one-at-a-time reference prior is

$$\pi_{r5}(\alpha,\beta,\phi) = \lim_{l\to\infty}\frac{\pi_1^l(\alpha,\beta,\phi)}{\pi_1^l(\alpha_0,\beta_0,\phi_0)} \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}, \tag{30}$$

where α_0, β_0 and ϕ_0 are an inner point of the interval $(0,\infty)$.

Lastly, we derive the one-at-a-time reference prior for the parameter grouping $\{\alpha,\phi,\beta\}$. For the derivation of the reference prior, we obtain the following quantities from the inverse matrix of the Fisher information (22).

$$h_1 = \frac{4+6\alpha+3\alpha^2}{\alpha^2(1+\alpha)^2(2+\alpha)^2}, h_2 = \frac{\alpha(2+\alpha)}{(1+\alpha)(3+\alpha)\phi^2} \text{ and } h_3 = \frac{1+\alpha}{(3+\alpha)\beta^2}.$$

Thus by the derivation method of the fifth reference prior for α, we can show that the one-at-a-time reference prior is

$$\pi_{r6}(\alpha,\phi,\beta) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}. \tag{31}$$

Appendix 5: Proof of Theorem 3.2

We consider the grouping orders $\{\beta,(\alpha,\phi)\}$, $\{(\alpha,\phi),\beta\}$, $\{\beta,\alpha,\phi\}$ and $\{\beta,\phi,\alpha\}$ in development of the reference priors for β. First, we derive the two group reference prior for the parameter grouping $\{\beta, (\alpha, \phi)\}$. For the derivation of the reference prior, from the Fisher information (21), we obtain determinant of the Fisher information (21) as follows.

$$|I(\alpha, \beta, \phi)| \;=\; \frac{4 + 6\alpha + 3\alpha^2}{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)^2\beta^2\phi^2}. \tag{32}$$

Also determinant of the Fisher information $I_2(\alpha, \phi)$ is given by

$$|I_2(\alpha, \phi)| \;=\; \frac{4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)\phi^2}. \tag{33}$$

Thus we have

$$h_1 = \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\beta^2}, h_2 = \frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)\phi^2}.$$

Step 1. Note that

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2} h_2^{1/2} d\alpha d\phi = \int_{a_3}^{b_3}\int_{a_2}^{b_2} \left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)\phi^2}\right]^{\frac{1}{2}} d\alpha d\phi = Q_1$$

It follows that

$$\pi_2^l(\alpha, \phi|\beta) = Q_1^{-1}\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)}\right]^{\frac{1}{2}} \phi^{-1}.$$

Step 2. Now we have

$$\begin{aligned} E^l\{\log h_1|\alpha, \phi\} &= \int_{a_3}^{b_3}\int_{a_2}^{b_2} \log\left(\frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\beta^2}\right)\pi_2^l(\alpha, \phi|\beta) d\alpha d\phi \\ &= Q_2 + \log\beta^{-2}. \end{aligned}$$

It follows that

$$\pi_1^l(\beta) \propto \exp[E^l\{\log h_1|\alpha, \phi\}/2] = \exp[Q_2/2]\beta^{-1}.$$

Therefore the two group reference prior is

$$\pi_{r1}(\beta, \alpha, \phi) = \lim_{l\to\infty} \frac{\pi_2^l(\alpha, \phi|\beta)\pi_1^l(\beta)}{\pi_2^l(\alpha_0, \phi_0|\beta_0)\pi_1^l(\beta_0)} \propto \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1}, \tag{34}$$

where $\alpha_0, \beta_0, \phi_0$ are an inner point of the interval $(0, \infty)$.

Second, we derive the two group reference prior for the parameter grouping $\{(\alpha, \phi), \beta\}$. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (21).

$$h_1 = \frac{4 + 6\alpha + 3\alpha^2}{\alpha(1+\alpha)^3(2+\alpha)(3+\alpha)\phi^2} \text{ and } h_2 = \frac{1+\alpha}{(3+\alpha)\beta^2}.$$

Then by the derivation method of the first reference prior for β, we can show that the two group reference prior is

$$\pi_{r2}(\alpha, \phi, \beta) \propto \frac{(4 + 6\alpha + 3\alpha^2)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)^{\frac{3}{2}}(2+\alpha)^{\frac{1}{2}}(3+\alpha)^{\frac{1}{2}}} \beta^{-1}\phi^{-1}. \tag{35}$$

Third, we derive the one-at-a-time reference prior for the parameter grouping $\{\beta, \alpha, \phi\}$. For the derivation of the reference prior, we obtain the following quantities from the inverse matrix of the Fisher information (22).

$$h_1 = \frac{\alpha(2+\alpha)(4 + 6\alpha + 3\alpha^2)}{(1+\alpha)(3+\alpha)(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)\beta^2},$$
$$h_2 = \frac{4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2} \text{ and } h_3 = \frac{1+\alpha}{(3+\alpha)\phi^2}.$$

Step 1. Note that

$$\int_{a_3}^{b_3} h_3^{1/2} d\phi = \int_{a_3}^{b_3} \left[\frac{1+\alpha}{(3+\alpha)\phi^2}\right]^{1/2} d\phi = \left[\frac{1+\alpha}{3+\alpha}\right]^{1/2} Q_1.$$

It follows that

$$\pi_3^l(\phi|\beta, \alpha) = \frac{h_3^{1/2}}{\int_{a_3}^{b_3} h_3^{1/2} d\phi} = Q_1^{-1}\phi^{-1}.$$

Step 2. Now we have

$$E^l\{\log h_2|\beta, \alpha\} = \int_{a_3}^{b_3} Q_1^{-1}\phi^{-1} \log\left[\frac{4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right] d\phi = \log\left[\frac{4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right].$$

It follows that

$$\int_{a_2}^{b_2} \exp[E^l\{\log h_2|\beta, \alpha\}/2] d\alpha = Q_2.$$

Hence

$$\pi_2^l(\alpha,\phi|\beta) = \frac{\pi_3^l(\phi|\beta,\alpha)\exp[E^l\{\log h_2|\beta,\alpha\}/2]}{\int_{a_2}^{b_2}\exp[E^l\{\log h_2|\beta,\alpha\}/2]d\alpha} = Q_1^{-1}Q_2^{-1}\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right]^{\frac{1}{2}}\phi^{-1}.$$

Step 3. In the final step,

$$\begin{aligned} E^l\{\log h_1|\beta\} &= \int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_1^{-1}Q_2^{-1}\phi^{-1}\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right]^{\frac{1}{2}} \\ &\times \log\left(\frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\beta^2}\right)d\phi d\alpha \\ &= Q_{31}+\log\beta^{-2}. \end{aligned}$$

It follows that

$$\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\beta\}/2]d\beta = \exp[Q_{31}/2]Q_3.$$

Hence

$$\begin{aligned} \pi_1^l(\beta,\alpha,\phi) &= \frac{\pi_2^l(\alpha,\phi|\beta)\exp[E^l\{\log h_1|\beta\}/2]}{\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\beta\}/2]d\beta} \\ &= Q_1^{-1}Q_2^{-1}Q_3^{-1}\frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}. \end{aligned}$$

Thus the one-at-a-time reference prior is

$$\pi_{r3}(\beta,\alpha,\phi) = \lim_{l\to\infty}\frac{\pi_1^l(\beta,\alpha,\phi)}{\pi_1^l(\beta_0,\alpha_0,\phi_0)} \propto \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}, \tag{36}$$

where α_0, β_0 and ϕ_0 are an inner point of the interval $(0,\infty)$.

Lastly, we derive the one-at-a-time reference prior for the parameter grouping $\{\beta,\phi,\alpha\}$. For the derivation of the reference prior, we obtain the following quantities from the inverse matrix of the Fisher information (22).

$$\begin{aligned} h_1 &= \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\beta^2}, \\ h_2 &= \frac{(1+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)}{(2+\alpha)^2(3+\alpha)(1+2\alpha+2\alpha^2)\phi^2} \text{ and } h_3 = \frac{1+2\alpha+2\alpha^2}{\alpha^2(1+\alpha)^2}. \end{aligned}$$

Thus by the derivation method of the third reference prior for β, we can show that the one-at-a-time reference prior is

$$\pi_{r4}(\alpha, \phi, \beta) \propto \frac{(1+2\alpha+2\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)} \beta^{-1}\phi^{-1}. \tag{37}$$

Appendix 6: Proof of Theorem 3.3

We consider the grouping orders $\{\phi, (\alpha, \beta)\}$, $\{(\alpha, \beta), \phi\}$, $\{\phi, \alpha, \beta\}$ and $\{\phi, \beta, \alpha\}$ in development of the reference priors for ϕ. First, we derive the two group reference prior for the parameter grouping $\{\phi, (\alpha, \beta)\}$. For the derivation of the reference prior, from the Fisher information (21), we obtain determinant of the Fisher information (21) as follows.

$$|I(\alpha, \beta, \phi)| = \frac{4+6\alpha+3\alpha^2}{\alpha(1+\alpha)^2(2+\alpha)(3+\alpha)^2\beta^2\phi^2}. \tag{38}$$

Also determinant of the Fisher information $I_2(\alpha, \beta)$ is given by

$$|I_2(\alpha, \beta)| = \frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)\beta^2}. \tag{39}$$

Thus we have

$$h_1 = \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\phi^2}, h_2 = \frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)\beta^2}.$$

Step 1. Note that

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2} h_2^{1/2} d\alpha d\beta = \int_{a_3}^{b_3}\int_{a_2}^{b_2} \left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)\beta^2}\right]^{\frac{1}{2}} d\alpha d\beta = Q_1$$

It follows that

$$\pi_2^l(\alpha, \beta|\phi) = Q_1^{-1}\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)(2+\alpha)^2(3+\alpha)}\right]^{\frac{1}{2}} \beta^{-1}.$$

Step 2. Now we have

$$\begin{aligned} E^l\{\log h_1|\alpha, \beta\} &= \int_{a_3}^{b_3}\int_{a_2}^{b_2} \log\left(\frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\phi^2}\right)\pi_2^l(\alpha, \beta|\phi) d\alpha d\beta \\ &= Q_2 + \log\phi^{-2}. \end{aligned}$$

It follows that

$$\pi_1^l(\phi) \propto \exp[E^l\{\log h_1|\alpha,\beta\}/2] = \exp[Q_2/2]\phi^{-1}.$$

Therefore the two group reference prior is

$$\pi_{r1}(\phi,\alpha,\beta) = \lim_{l\to\infty} \frac{\pi_2^l(\alpha,\beta|\phi)\pi_1^l(\phi)}{\pi_2^l(\alpha_0,\beta_0|\phi_0)\pi_1^l(\phi_0)} \propto \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1}, \tag{40}$$

where $\alpha_0, \beta_0, \phi_0$ are an inner point of the interval $(0,\infty)$.

Second, we derive the two group reference prior for the parameter grouping $\{(\alpha,\beta),\phi\}$. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (21).

$$h_1 = \frac{4+6\alpha+3\alpha^2}{\alpha(1+\alpha)^3(2+\alpha)(3+\alpha)\beta^2} \text{ and } h_2 = \frac{1+\alpha}{(3+\alpha)\phi^2}.$$

Then by the derivation method of the first reference prior for ϕ, we can show that the two group reference prior is

$$\pi_{r2}(\alpha,\phi,\beta) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)^{\frac{3}{2}}(2+\alpha)^{\frac{1}{2}}(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1}. \tag{41}$$

Third, we derive the one-at-a-time reference prior for the parameter grouping $\{\phi,\alpha,\beta\}$. For the derivation of the reference prior, we obtain the following quantities from the inverse matrix of the Fisher information (22).

$$h_1 = \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\phi^2},$$
$$h_2 = \frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2} \text{ and } h_3 = \frac{1+\alpha}{(3+\alpha)\beta^2}.$$

Step 1. Note that

$$\int_{a_3}^{b_3} h_3^{1/2} d\beta = \int_{a_3}^{b_3}\left[\frac{1+\alpha}{(3+\alpha)\beta^2}\right]^{1/2} d\beta = \left[\frac{1+\alpha}{3+\alpha}\right]^{1/2} Q_1.$$

It follows that

$$\pi_3^l(\beta|\phi,\alpha) = \frac{h_3^{1/2}}{\int_{a_3}^{b_3} h_3^{1/2} d\beta} = Q_1^{-1}\beta^{-1}.$$

Step 2. Now we have

$$E^l\{\log h_2|\phi,\alpha\} = \int_{a_3}^{b_3} Q_1^{-1}\beta^{-1}\log\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right]d\beta = \log\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right].$$

It follows that

$$\int_{a_2}^{b_2} \exp[E^l\{\log h_2|\phi,\alpha\}/2]d\alpha = Q_2.$$

Hence

$$\pi_2^l(\alpha,\beta|\phi) = \frac{\pi_3^l(\beta|\phi,\alpha)\exp[E^l\{\log h_2|\phi,\alpha\}/2]}{\int_{a_2}^{b_2}\exp[E^l\{\log h_2|\phi,\alpha\}/2]d\alpha} = Q_1^{-1}Q_2^{-1}\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right]^{\frac{1}{2}}\beta^{-1}.$$

Step 3. In the final step,

$$\begin{aligned} E^l\{\log h_1|\phi\} &= \int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_1^{-1}Q_2^{-1}\beta^{-1}\left[\frac{4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4}{\alpha^2(1+\alpha)^2(2+\alpha)^2}\right]^{\frac{1}{2}} \\ &\times \log\left(\frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\phi^2}\right)d\beta d\alpha \\ &= Q_{31}+\log\phi^{-2}. \end{aligned}$$

It follows that

$$\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\phi\}/2]d\phi = \exp[Q_{31}/2]Q_3.$$

Hence

$$\begin{aligned} \pi_1^l(\phi,\alpha,\beta) &= \frac{\pi_2^l(\alpha,\beta|\phi)\exp[E^l\{\log h_1|\phi\}/2]}{\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\phi\}/2]d\phi} \\ &= Q_1^{-1}Q_2^{-1}Q_3^{-1}\frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}. \end{aligned}$$

Thus the one-at-a-time reference prior is

$$\pi_{r3}(\phi,\alpha,\beta) = \lim_{l\to\infty}\frac{\pi_1^l(\phi,\alpha,\beta)}{\pi_1^l(\phi_0,\alpha_0,\beta_0)} \propto \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}, \qquad (42)$$

where α_0,β_0 and ϕ_0 are an inner point of the interval $(0,\infty)$.

Lastly, we derive the one-at-a-time reference prior for the parameter grouping $\{\phi, \beta, \alpha\}$. For the derivation of the reference prior, we obtain the following quantities from the inverse matrix of the Fisher information (22).

$$\begin{aligned} h_1 &= \frac{\alpha(2+\alpha)(4+6\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)\phi^2}, \\ h_2 &= \frac{(1+\alpha)(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)}{(2+\alpha)^2(3+\alpha)(1+2\alpha+2\alpha^2)\beta^2} \text{ and } h_3 = \frac{1+2\alpha+2\alpha^2}{\alpha^2(1+\alpha)^2}. \end{aligned}$$

Thus by the derivation method of the third reference prior for ϕ, we can show that the one-at-a-time reference prior is

$$\pi_{r4}(\alpha, \phi, \beta) \propto \frac{(1+2\alpha+2\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)}\beta^{-1}\phi^{-1}. \tag{43}$$

Appendix 7: Proof of Theorem 4.1

Proof. Under the class of priors, the joint posterior for (α, β, ϕ) given $\mathbf{z}$ is

$$\begin{aligned} \pi(\alpha, \beta, \phi|\mathbf{z}) &\propto \frac{(4+6\alpha+3\alpha^2)^{a_1}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4-n}(1+\alpha)^{a_5-n}(2+\alpha)^{a_6}(3+\alpha)^{a_7}} \\ &\times \beta^{n-b}\phi^{n-c}\left[\prod_{i=1}^{n}(1+\beta x_i+\phi y_i)^{-(\alpha+2)}\right], \end{aligned} \tag{44}$$

where $\mathbf{z} = ((x_1, y_1), \cdots, (x_n, y_n))$. Then

$$\begin{aligned} \pi(\alpha, \beta, \phi|\mathbf{z}) &\propto \frac{(4+6\alpha+3\alpha^2)^{a_1}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4-n}(1+\alpha)^{a_5-n}(2+\alpha)^{a_6}(3+\alpha)^{a_7}} \\ &\times \beta^{n-b}\phi^{n-c}(1+\beta x_i+\phi y_i)^{-n(\alpha+2)}. \end{aligned} \tag{45}$$

Integrating with respect to ϕ for the posterior (45), then we obtain

$$\begin{aligned} \pi(\alpha, \beta|\mathbf{z}) &\propto \frac{(4+6\alpha+3\alpha^2)^{a_1}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4-n}(1+\alpha)^{a_5-n}(2+\alpha)^{a_6}(3+\alpha)^{a_7}} \\ &\times \frac{\Gamma[n-c+1]\Gamma[n\alpha+n+c-1]}{\Gamma[n(2+\alpha)]}\beta^{n-b}(1+\beta x_i)^{-n(\alpha+1)-c+1}, \end{aligned} \tag{46}$$

if $n-c+1>0$ and $n\alpha+n+c-1>0$. Integrating with respect to β for the posterior (46), then we obtain

$$\begin{aligned}\pi(\alpha|\mathbf{z}) &\propto \frac{\Gamma[n-b+1]\Gamma[n\alpha+b+c-2]}{\Gamma[n(2+\alpha)]} \\ &\times \frac{(4+6\alpha+3\alpha^2)^{a_1}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4-n}(1+\alpha)^{a_5-n}(2+\alpha)^{a_6}(3+\alpha)^{a_7}} \\ &\propto \frac{\Gamma[n\alpha+b+c-2]}{\Gamma[n(2+\alpha)]} \\ &\times \frac{(4+6\alpha+3\alpha^2)^{a_1}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4-n}(1+\alpha)^{a_5-n}(2+\alpha)^{a_6}(3+\alpha)^{a_7}},\end{aligned} \tag{47}$$

if $n-b+1>0$ and $n\alpha+b+c-2>0$. Then for $0<\alpha\le 1$, we have

$$\pi(\alpha|\mathbf{z}) \le \frac{\Gamma[n+b+c-2]}{\Gamma[2n]}\frac{2^n 13^{a_1}37^{a_2}5^{a_3}}{\alpha^{a_4-n}2^{a_6}3^{a_7}} \equiv \pi'(\alpha|\mathbf{z}) \tag{48}$$

Thus

$$\int_0^1 \pi'(\alpha|\mathbf{z})d\alpha < \infty, \tag{49}$$

if $n-a_4+1>0$. Next for $1<\alpha<\infty$, we have

$$\begin{aligned}\pi(\alpha|\mathbf{z}) &\propto \frac{\Gamma[n\alpha+b+c-2]}{\Gamma[n(2+\alpha)]} \\ &\times \frac{(4+6\alpha+3\alpha^2)^{a_1}(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4-n}(1+\alpha)^{a_5-n}(2+\alpha)^{a_6}(3+\alpha)^{a_7}} \\ &\propto \frac{\Gamma[n\alpha+b+c-2]}{\Gamma[n(2+\alpha)]}\alpha^n(1+\alpha)^n\alpha^{2a_1+4a_2+2a_3-a_4-a_5-a_6-a_7} \\ &\times \frac{(\frac{4}{\alpha^2}+\frac{6}{\alpha}+3)^{a_1}(\frac{4}{\alpha^4}+\frac{12}{\alpha^3}+\frac{14}{\alpha^2}+\frac{6}{\alpha}+1)^{a_2}(\frac{1}{\alpha^2}+\frac{2}{\alpha}+2)^{a_3}}{(\frac{1}{\alpha}+1)^{a_5}(\frac{2}{\alpha}+1)^{a_6}(\frac{3}{\alpha}+1)^{a_7}} \\ &\le 13^{a_1}37^{a_2}5^{a_3}\frac{\Gamma[n\alpha+b+c-2]}{\Gamma[n(2+\alpha)]}\alpha^n(1+\alpha)^n\alpha^{2a_1+4a_2+2a_3-a_4-a_5-a_6-a_7}.\end{aligned} \tag{50}$$

Now for $1<\alpha<\infty$, we have the following inequality.

$$\begin{aligned}\frac{\Gamma[n\alpha+b+c-2]}{\Gamma[n(2+\alpha)]}\frac{\alpha^n(1+\alpha)^n}{\alpha^{b+c-2}} &= \frac{\Gamma[n\alpha+b+c-2]}{\Gamma[n\alpha]\prod_{i=1}^{2n}(n\alpha+2n-i)}\frac{\alpha^n(1+\alpha)^n}{\alpha^{b+c-2}} \\ &= \frac{\Gamma[n\alpha+b+c-2]}{\alpha^{b+c-2}\Gamma[n\alpha]}\frac{\left(1+\frac{1}{\alpha}\right)^n}{\prod_{i=1}^{2n}[n(1+\frac{2}{\alpha})-\frac{i}{\alpha}]} < \infty,\end{aligned} \tag{51}$$

since $\Gamma[n\alpha]$ is approximately equal to $\sqrt{2\pi(n\alpha-1)}(n\alpha-1)^{n\alpha-1}e^{-(n\alpha-1)}$ for large α. Therefore we have

$$\pi(\alpha|\mathbf{z}) \le k_1\alpha^{2a_1+4a_2+2a_3-a_4-a_5-a_6-a_7+b+c-2} \equiv \pi'(\alpha|\mathbf{z}), \tag{52}$$

where k_1 is a constant. Thus

$$\int_1^\infty \pi'(\alpha|\mathbf{z})d\alpha < \infty, \tag{53}$$

if $a_4 + a_5 + a_6 + a_7 - 2a_1 - 4a_2 - 2a_3 - b - c + 1 > 0$. This completes the proof. $\square$

Appendix 8: Proof of Theorem 4.2

First, we consider the one-at-a-time reference prior (**??**) for β and given by

$$\pi(\beta, \alpha, \phi) \propto \frac{(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}. \tag{54}$$

Then the joint posterior of (α, β, ϕ) is

$$\begin{aligned}\pi(\alpha, \beta, \phi|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)^{\frac{1}{2}}}{(1+\alpha)(2+\alpha)}\beta^{n-1}\phi^{n-1} \\ &\times \left[\prod_{i=1}^{n}(1 + \beta x_i + \phi y_i)^{-(\alpha+2)}\right],\end{aligned} \tag{55}$$

where $\mathbf{z} = ((x_1, y_1), \cdots, (x_n, y_n))$. Thus

$$\begin{aligned}\pi(\alpha, \beta, \phi|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)^{\frac{1}{2}}}{(1+\alpha)(2+\alpha)}\beta^{n-1}\phi^{n-1} \\ &\times (1 + \beta x_i + \phi y_i)^{-n(\alpha+2)}.\end{aligned} \tag{56}$$

Integrating with respect to ϕ for the posterior (56), then we obtain

$$\begin{aligned}\pi(\alpha, \beta|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)^{\frac{1}{2}}}{(1+\alpha)(2+\alpha)} \frac{\Gamma[n(1+\alpha)]}{\Gamma[n(2+\alpha)]} \\ &\times \beta^{n-1}(1 + \beta x_i)^{-n(\alpha+1)}.\end{aligned} \tag{57}$$

Integrating with respect to β for the posterior (57), then we obtain

$$\begin{aligned}\pi(\alpha|\mathbf{z}) &\propto \frac{(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)^{\frac{1}{2}}}{(1+\alpha)(2+\alpha)} \frac{\Gamma[n\alpha]}{\Gamma[n(2+\alpha)]}\alpha^{n-1}(1+\alpha)^n \\ &\propto \frac{(4 + 12\alpha + 14\alpha^2 + 6\alpha^3 + \alpha^4)^{\frac{1}{2}}}{(1+\alpha)(2+\alpha)} \frac{\alpha^{n-1}(1+\alpha)^n}{\prod_{i=1}^{2n}(n\alpha + 2n - i)}.\end{aligned} \tag{58}$$

Thus

$$\begin{aligned}
\int_1^\infty \pi(\alpha|\mathbf{z})d\alpha &\propto \int_1^\infty \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)(2+\alpha)} \frac{\alpha^{n-1}(1+\alpha)^n}{\prod_{i=1}^{2n}(n\alpha+2n-i)}d\alpha \\
&\geq \int_1^\infty \frac{\alpha^{2n-1}}{6\alpha^{2n}\prod_{i=1}^{2n}(n+\frac{2n-i}{\alpha})}d\alpha \\
&= \int_1^\infty \frac{\alpha^{-1}}{6\prod_{i=1}^{2n}(n+\frac{2n-i}{\alpha})}d\alpha \\
&\geq \int_1^\infty \frac{\alpha^{-1}}{6\prod_{i=1}^{2n}(3n-i)}d\alpha = \infty. \qquad (59)
\end{aligned}$$

Thus the posterior distribution is improper.

Second, we consider the two group reference prior (**??**) for β and given by

$$\pi(\beta,\alpha,\phi) \propto \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{\alpha(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1}. \qquad (60)$$

Then the joint posterior of (α,β,ϕ) is

$$\begin{aligned}
\pi(\alpha,\beta,\phi|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}} \\
&\times \beta^{n-1}\phi^{n-1}\left[\prod_{i=1}^n (1+\beta x_i+\phi y_i)^{-(\alpha+2)}\right], \qquad (61)
\end{aligned}$$

where $\mathbf{z} = ((x_1,y_1),\cdots,(x_n,y_n))$. Then

$$\begin{aligned}
\pi(\alpha,\beta,\phi|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}} \\
&\times \beta^{n-1}\phi^{n-1}(1+\beta x_i+\phi y_i)^{-n(\alpha+2)}. \qquad (62)
\end{aligned}$$

Integrating with respect to ϕ for the posterior (62), then we obtain

$$\begin{aligned}
\pi(\alpha,\beta|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}} \\
&\times \frac{\Gamma[n(1+\alpha)]}{\Gamma[n(2+\alpha)]}\beta^{n-1}(1+\beta x_i)^{-n(\alpha+1)}. \qquad (63)
\end{aligned}$$

Integrating with respect to β for the posterior (63), then we obtain

$$\begin{aligned}
\pi(\alpha|\mathbf{z}) &\propto \alpha^{n-1}(1+\alpha)^n \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}} \frac{\Gamma[n\alpha]}{\Gamma[n(2+\alpha)]} \\
&\propto \alpha^{n-1}(1+\alpha)^n \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}} \frac{1}{\prod_{i=1}^{2n}(n\alpha+2n-i)}. \qquad (64)
\end{aligned}$$

Thus

$$\begin{aligned}
\int_1^\infty \pi(\alpha|\mathbf{z})d\alpha &\propto \int_1^\infty \alpha^{n-1}(1+\alpha)^n \frac{(4+12\alpha+14\alpha^2+6\alpha^3+\alpha^4)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}(2+\alpha)(3+\alpha)^{\frac{1}{2}}} \frac{1}{\prod_{i=1}^{2n}(n\alpha+2n-i)} d\alpha \\
&\geq \int_1^\infty \frac{\alpha^{2n-1}}{\sqrt{72}\alpha^{2n}\prod_{i=1}^{2n}(n+\frac{2n-i}{\alpha})} d\alpha \\
&= \int_1^\infty \frac{\alpha^{-1}}{\sqrt{72}\prod_{i=1}^{2n}(n+\frac{2n-i}{\alpha})} d\alpha \\
&\geq \int_1^\infty \frac{\alpha^{-1}}{\sqrt{72}\prod_{i=1}^{2n}(3n-i)} d\alpha = \infty. \qquad (65)
\end{aligned}$$

Thus the posterior distribution is improper. Finally, using the above proof methods, we can show that the posterior under the remaining reference prior is improper. This completes the proof. □