SUPPLEMENTARY MATERIAL FOR THE RATIONAL SPDE APPROACH FOR GAUSSIAN RANDOM FIELDS WITH GENERAL SMOOTHNESS

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Appendix A. Iterated finite element method

The rational approximation $u_{h,m}^R$ of the solution u to (1.2) introduced in §3.3 is defined in terms of the discrete operators $P_{\ell,h} = p_\ell(L_h)$ and $P_{r,h} = p_r(L_h)$ via (3.5). Since the differential operator L in (3.1) is of second order, their continuous counterparts $P_\ell = p_\ell(L)$ and $P_r = p_r(L)$ in (3.7) are differential operators of order $2(m + m_\beta)$ and $2m$, respectively. Using a standard Galerkin approach for solving (3.7) would therefore require finite element basis functions $\{\varphi_j\}$ in the Sobolev space $H^{m+m_\beta}(\mathcal{D})$, which are difficult to construct in more than one space dimension. This can be avoided by using a modified version of the iterated Hilbert space approximation method by Lindgren et al. (2011), and in this section we give the details of this procedure.

Recall from §3.2 that $V_h \subset V$ is a finite element space with continuous piecewise linear basis functions $\{\varphi_j\}_{j=1}^{n_h}$ defined with respect to a regular triangulation $\mathcal{T}_h$ of the domain $\overline{\mathcal{D}}$ with mesh width $h := \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$.

For computing the finite element approximation, we start by factorizing the polynomials q_1 and q_2 in the rational approximation $\hat{r}$ of $\hat{f}(x) = x^{\beta - m_\beta}$ in terms of their roots,

$$q_1(x) = \sum_{i=1}^{m} c_i x^i = c_m \prod_{i=1}^{m} (x - r_{1i}) \quad \text{and} \quad q_2(x) = \sum_{j=1}^{m+1} b_j x^j = b_{m+1} \prod_{j=1}^{m+1} (x - r_{2j}).$$

We use these expressions to reformulate (3.9) as

$$x^{-\beta} = f(x^{-1}) \approx \hat{r}(x^{-1}) x^{-m_\beta} = \frac{c_m \prod_{i=1}^{m}(1 - r_{1i}x)}{b_{m+1} x^{m_\beta - 1} \prod_{j=1}^{m+1}(1 - r_{2i}x)},$$

where, again, we have expanded the fraction with x^m. This representation shows that we can equivalently define the rational SPDE approximation $u_{h,m}^R$ as the solution to (3.5) with $P_{\ell,h}, P_{r,h}$ redefined as $P_{\ell,h} = b_{m+1} L_h^{m_\beta - 1} \prod_{j=1}^{m+1} (\mathrm{Id}_h - r_{2j} L_h)$ and $P_{r,h} = c_m \prod_{i=1}^{m} (\mathrm{Id}_h - r_{1i} L_h)$, where Id_h denotes the identity on V_h.

We use the formulation of (3.5) as a system outlined in (3.6): First we solve $P_{\ell,h} x_{h,m} = \mathcal{W}_h$ and we then compute $u_{h,m}^R = P_{r,h} x_{h,m}$. To this end, we define the functions $x_k \in L_2(\Omega; V_h)$ for $k \in \{1, \ldots, m+m_\beta\}$ iteratively by

$$\begin{aligned} b_{m+1}(\mathrm{Id}_h - r_{21}L_h)x_1 &= \mathcal{W}_h, \\ (\mathrm{Id}_h - r_{2k}L_h)x_k &= x_{k-1}, \qquad k = 2, \ldots, m+1, \\ L_h x_k &= x_{k-1}, \qquad k = m+2, \ldots, m+m_\beta, \quad \text{if } m_\beta \geq 2, \end{aligned}$$

noting that $x_{m+m_\beta} = x_{h,m}$.

By recalling the bilinear form a_L from (3.2) and expanding $x_k = \sum_{j=1}^{n_h} x_{kj}\varphi_j$ with respect to the finite element basis, we find that the stochastic weights $\mathbf{x}_k = (x_{k1}, \ldots, x_{kn_h})^\top$ satisfy

$$\sum_{j=1}^{n_h} x_{1j}\, b_{m+1}\big((\varphi_j, \varphi_i)_{L_2(\mathcal{D})} - r_{21}\, a_L(\varphi_j, \varphi_i)\big) = (\mathcal{W}_h, \varphi_i)_{L_2(\mathcal{D})},$$

$$\sum_{j=1}^{n_h} x_{kj}\big((\varphi_j, \varphi_i)_{L_2(\mathcal{D})} - r_{2k}\, a_L(\varphi_j, \varphi_i)\big) = \sum_{j=1}^{n_h} x_{k-1,j}\,(\varphi_j, \varphi_i)_{L_2(\mathcal{D})},\ 2 \leq k \leq m+1,$$

$$\sum_{j=1}^{n_h} x_{kj}\, a_L(\varphi_j, \varphi_i) = \sum_{j=1}^{n_h} x_{k-1,j}\,(\varphi_j, \varphi_i)_{L_2(\mathcal{D})}, \quad k = m+2, \ldots, m+m_\beta,$$

where each of these equations hold for $i = 1, \ldots, n_h$. Recall from §3.2 that $\mathcal{W}_h$ is white noise in V_h. This entails the distribution $\big((\mathcal{W}_h, \varphi_i)_{L_2(\mathcal{D})}\big)_{i=1}^{n_h} \sim \mathsf{N}(\mathbf{0}, \mathbf{C})$, where $\mathbf{C}$ is the mass matrix with elements $C_{ij} = (\varphi_j, \varphi_i)_{L_2(\mathcal{D})}$ and, therefore, $\mathbf{x}_k \sim \mathsf{N}\big(\mathbf{0}, \mathbf{P}_{\ell,k}^{-1}\mathbf{C}\mathbf{P}_{\ell,k}^{-\top}\big)$ for every $k \in \{1, ..., m+m_\beta\}$. Here, the matrix $\mathbf{P}_{\ell,k}$ is defined by

$$\mathbf{P}_{\ell,k} = \begin{cases} b_{m+1}\mathbf{C}\,\mathbf{L}_k, & k = 1, \ldots, m+1, \\ b_{m+1}\mathbf{C}\left(\mathbf{C}^{-1}\mathbf{L}\right)^{k-m-1}\mathbf{L}_{m+1}, & k = m+2, \ldots, m+m_\beta, \end{cases}$$

where $\mathbf{L}_k := \prod_{j=1}^{k}\left(\mathbf{I} - r_{2j}\mathbf{C}^{-1}\mathbf{L}\right)$, with identity matrix $\mathbf{I} \in \mathbb{R}^{n_h \times n_h}$, and the entries of $\mathbf{L}$ are given by

$$L_{ij} := a_L(\varphi_j, \varphi_i) = (\mathbf{H}\nabla\varphi_j, \nabla\varphi_i)_{L_2(\mathcal{D})} + \left(\kappa^2\varphi_j, \varphi_i\right)_{L_2(\mathcal{D})}, \qquad i, j = 1, \ldots, n_h,$$

cf. (3.1)–(3.2). In particular, the weights $\mathbf{x}$ of $x_{h,m}$ have distribution

$$\mathbf{x} \sim \mathsf{N}\left(\mathbf{0}, \mathbf{P}_\ell^{-1}\mathbf{C}\mathbf{P}_\ell^{-\top}\right), \qquad \text{where} \qquad \mathbf{P}_\ell := \mathbf{P}_{\ell, m+m_\beta}. \tag{A.1}$$

Note also that for the Matérn case, i.e., $L = \kappa^2 - \Delta$, we have $\mathbf{L} = \kappa^2\mathbf{C} + \mathbf{G}$, where $\mathbf{G}$ is the stiffness matrix with elements $G_{ij} = (\nabla\varphi_j, \nabla\varphi_i)_{L_2(\mathcal{D})}$.

To calculate the final approximation $u_{h,m}^R = P_{r,h}x_{h,m}$, we apply a similar iterative procedure. Let $u_1, \ldots, u_m$ be defined by

$$\begin{aligned} u_1 &= c_m(\mathrm{Id}_h - r_{11}L_h)x_{h,m}, \\ u_k &= (\mathrm{Id}_h - r_{1k}L_h)u_{k-1}, \qquad k = 2, \ldots, m. \end{aligned}$$

Then $u_{h,m}^R = c_m\big(\prod_{i=1}^{m}(\mathrm{Id} - r_{1i}L_h)\big)x_{h,m} = u_m$ and the weights $\mathbf{u}_k$ of u_k can be obtained from the weights of $x_{h,m}$ via $\mathbf{u}_k = \mathbf{P}_{r,k}\,\mathbf{x}$, where

$$\mathbf{P}_{r,k} := c_m \prod_{i=1}^{k}\left(\mathbf{I} - r_{1i}\mathbf{C}^{-1}\mathbf{L}\right).$$

By (A.1), the distribution of the weights $\mathbf{u}$ of the final rational approximation $u_{h,m}^R$ is thus given by

$$\mathbf{u} \sim \mathsf{N}\left(\mathbf{0}, \mathbf{P}_r \mathbf{P}_\ell^{-1} \mathbf{C} \mathbf{P}_\ell^{-\top} \mathbf{P}_r^\top\right), \qquad \text{where} \qquad \mathbf{P}_r := \mathbf{P}_{r,m}.$$

To obtain sparse matrices $\mathbf{P}_\ell$ and $\mathbf{P}_r$, we approximate the mass matrix $\mathbf{C}$ by a diagonal matrix $\widetilde{\mathbf{C}}$ with diagonal elements $\widetilde{C}_{ii} = \sum_{j=1}^{n_h} C_{ij}$. The effect of this "mass lumping" was motivated theoretically by Lindgren et al. (2011), and was empirically shown to be small by Bolin and Lindgren (2013).

Appendix B. Convergence analysis

In this section we give the details of the convergence result stated in Theorem 3.3. As mentioned in §3.4, we choose $\hat{r} = \hat{r}_h$ as the L_∞-best rational approximation of $\hat{f}(x) = x^{\beta - m_\beta}$ on the interval J_h for each h. We furthermore assume that the operator L in (3.1) is normalized such that $\lambda_1 \geq 1$ and, thus, $J_h \subset J \subset [0,1]$.

Recall that Proposition 3.2 provides a bound for $\|u - u_h\|_{L_2(\Omega;L_2(\mathcal{D}))}$. Therefore, it remains is to estimate the strong error between $u_{h,m}^R$ and u_h induced by the rational approximation of $f(x) = x^\beta$. To this end, recall the construction of the rational approximation $u_{h,m}^R$ from §3.3: We first decomposed f as $f(x) = \hat{f}(x) x^{m_\beta}$, where $\hat{f}(x) = x^{\beta - m_\beta}$, and then used a rational approximation $\hat{r} = \frac{q_1}{q_2}$ of $\hat{f}$ on the interval $J_h = \left[\lambda_{n_h,h}^{-1}, \lambda_{1,h}^{-1}\right]$ with $q_1 \in \mathcal{P}^m(J_h)$ and $q_2 \in \mathcal{P}^{m+1}(J_h)$ to define the approximation $r(x) := \hat{r}(x) x^{m_\beta}$ of f. Here, $\mathcal{P}^m(J_h)$ denotes the set of polynomials $q \colon J_h \to \mathbb{R}$ of degree $\deg(q) = m$. In the following, we assume that $\hat{r} = \hat{r}_h$ is the best rational approximation of $\hat{f}$ of this form, i.e.,

$$\|\hat{f} - \hat{r}_h\|_{C(J_h)} = \inf\left\{ \|\hat{f} - \hat{\rho}\|_{C(J_h)} : \hat{\rho} = \tfrac{q_1}{q_2},\ q_1 \in \mathcal{P}^m(J_h),\ q_2 \in \mathcal{P}^{m+1}(J_h) \right\},$$

where $\|g\|_{C(J)} := \sup_{x \in J} |g(x)|$.

For the analysis, we treat the two cases $\beta \in (0,1)$ and $\beta \geq 1$ separately. If $\beta \geq 1$, then $\hat{\beta} := \beta - m_\beta \in [0,1)$. Thus, if $\hat{r}_*$ denotes the best rational approximation of $\hat{f}$ on the interval $[0,1]$, we find (Stahl, 2003, Theorem 1)

$$\|\hat{f} - \hat{r}_h\|_{C(J_h)} \leq \sup_{x \in [0,1]} |\hat{f}(x) - \hat{r}_*(x)| \leq \hat{C} e^{-2\pi\sqrt{\hat{\beta} m}},$$

where the constant $\hat{C} > 0$ is continuous in $\hat{\beta}$ and independent of h and the degree m. Since $x^{m_\beta} \leq 1$ for all $x \in J_h$, we obtain for $r_h(x) := \hat{r}_h(x) x^{m_\beta}$ the same bound,

$$\|f - r_h\|_{C(J_h)} \leq \sup_{x \in J_h} |\hat{f}(x) - \hat{r}_h(x)| \leq \hat{C} e^{-2\pi\sqrt{\hat{\beta} m}}. \tag{B.1}$$

If $\beta \in (0,1)$, then $\hat{\beta} \in (-1,0)$ and we let $\widetilde{r}$ be the best approximation of $\widetilde{f}(x) := x^{|\hat{\beta}|}$ on $[0,1]$. A rational approximation of $\widetilde{f}$ on the different interval $\widetilde{J}_h := [\lambda_{1,h}, \lambda_{n_h,h}]$ is then given by $\widetilde{R}_h(\widetilde{x}) := \lambda_{n_h,h}^{|\hat{\beta}|} \widetilde{r}(\lambda_{n_h,h}^{-1} \widetilde{x})$ with error

$$\sup_{\widetilde{x} \in \widetilde{J}_h} |\widetilde{f}(\widetilde{x}) - \widetilde{R}_h(\widetilde{x})| \leq \lambda_{n_h,h}^{|\hat{\beta}|} \sup_{x \in [0,1]} |\widetilde{f}(x) - \widetilde{r}(x)| \leq \widetilde{C} \lambda_{n_h,h}^{|\hat{\beta}|} e^{-2\pi\sqrt{|\hat{\beta}| m}},$$

where the constant $\widetilde{C} > 0$ depends only on $|\hat{\beta}|$. On $J_h = \left[\lambda_{n_h,h}^{-1}, \lambda_{1,h}^{-1}\right]$ the function $\widetilde{R}_h(x^{-1})$ is an approximation of $\hat{f}(x) = x^{\hat{\beta}} = \widetilde{f}(x^{-1})$ and

$$\|\hat{f} - \hat{r}_h\|_{C(J_h)} \leq \sup_{x \in J_h} |\hat{f}(x) - \widetilde{R}_h(x^{-1})| \leq \sup_{\widetilde{x} \in \widetilde{J}_h} |\widetilde{f}(\widetilde{x}) - \widetilde{R}_h(\widetilde{x})| \leq \widetilde{C} \lambda_{n_h,h}^{|\hat{\beta}|} e^{-2\pi\sqrt{|\hat{\beta}|m}}.$$

Finally, we use again the estimate $x^{m_\beta} \leq 1$ on J_h to derive

$$\|f - r_h\|_{C(J_h)} \leq \|\hat{f} - \hat{r}_h\|_{C(J_h)} \leq \widetilde{C} \lambda_{n_h,h}^{|\hat{\beta}|} e^{-2\pi\sqrt{|\hat{\beta}|m}}. \tag{B.2}$$

Proposition 3.2 and the estimates (B.1)–(B.2) yield Theorem 3.3, which is proven below.

Proof of Theorem 3.3. By Proposition 3.2, it suffices to bound $\mathbb{E}\|u_h - u_{h,m}^R\|_{L_2(\mathcal{D})}^2$. To this end, let $\mathcal{W}_h = \sum_{j=1}^{n_h} \xi_j e_{j,h}$ be a Karhunen–Loève expansion of $\mathcal{W}_h$, where $\{e_{j,h}\}_{j=1}^{n_h}$ are $L_2(\mathcal{D})$-orthonormal eigenvectors of L_h corresponding to the eigenvalues $\{\lambda_{j,h}\}_{j=1}^{n_h}$ and $\xi_j \sim \mathsf{N}(0,1)$ i.i.d..

By construction and owing to boundedness and invertibility of L_h, we have for $u_{h,m}^R$ in (3.5) that $u_{h,m}^R = P_{\ell,h}^{-1} P_{r,h} \mathcal{W}_h = r_h(L_h^{-1}) \mathcal{W}_h$ and we estimate

$$\mathbb{E}\|u_h - u_{h,m}^R\|_{L_2(\mathcal{D})}^2 = \mathbb{E} \sum_{j=1}^{n_h} \xi_j^2 \left(\lambda_{j,h}^{-\beta} - r_h(\lambda_{j,h}^{-1})\right)^2 \leq n_h \max_{1 \leq j \leq n_h} \left|\lambda_{j,h}^{-\beta} - r_h(\lambda_{j,h}^{-1})\right|^2.$$

By (B.1) and (B.2), we can bound the last term by

$$\max_{1 \leq j \leq n_h} \left|\lambda_{j,h}^{-\beta} - r_h(\lambda_{j,h}^{-1})\right|^2 \leq \left(\sup_{x \in J_h} |f(x) - r_h(x)|\right)^2 \lesssim \lambda_{n_h,h}^{2\max\{(1-\beta),0\}} e^{-4\pi\sqrt{|\beta - m_\beta|m}}.$$

By (Strang and Fix, 2008, Theorem 6.1) we have $\lambda_{n_h,h} \lesssim \lambda_{n_h} \lesssim n_h^{2/d}$, for sufficiently small $h \in (0,1)$, where the last bound follows from the Weyl asymptotic (3.3). Finally, $n_h \lesssim h^{-d}$ by quasi-uniformity of the triangulation $\mathcal{T}_h$. Thus, we conclude

$$\mathbb{E}\|u_h - u_{h,m}^R\|_{L_2(\mathcal{D})}^2 \lesssim h^{-4\max\{(1-\beta),\,0\}-d} e^{-4\pi\sqrt{|\beta - m_\beta|m}},$$

which combined with Proposition 3.2 proves Theorem 3.3. □

Appendix C. A comparison to the quadrature approach

Bolin et al. (2018) proposed another method which can be applied to simulate the solution u to (1.2) numerically. The approach therein is to express the discretized equation (3.4) as $L_h^{\tilde{\beta}} L_h^{\lfloor\beta\rfloor} u_h = \mathcal{W}_h$, where $\tilde{\beta} = \beta - \lfloor\beta\rfloor \in [0,1)$. Since $L_h^{\lfloor\beta\rfloor} u_h = f$ can be solved by using non-fractional methods, the focus was on the fractional case $\beta \in (0,1)$ when constructing the approximative solution. From the Dunford–Taylor calculus (Yosida, 1995, §IX.11) one has in this case the following representation of the discrete inverse,

$$L_h^{-\beta} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \lambda^{-\beta} \left(\lambda \operatorname{Id}_h + L_h\right)^{-1} \mathrm{d}\lambda.$$

Bonito and Pasciak (2015) introduced a quadrature approximation $Q_{h,k}^\beta$ of this integral after a change of variables $\lambda = e^{-2y}$ and based on an equidistant grid for

y with step size $k > 0$, i.e.,

$$Q^{\beta}_{h,k} := \frac{2k\sin(\pi\beta)}{\pi} \sum_{j=-K^-}^{K^+} e^{2\beta y_j} \left(\mathrm{Id}_h + e^{2y_j} L_h\right)^{-1}, \quad \text{where} \quad y_j := jk.$$

Exponential convergence of order $\mathcal{O}\big(e^{-\pi^2/(2k)}\big)$ of the operator $Q^{\beta}_{h,k}$ to the discrete fractional inverse $L_h^{-\beta}$ was proven for $K^- := \left\lceil \frac{\pi^2}{4\beta k^2} \right\rceil$ and $K^+ := \left\lceil \frac{\pi^2}{4(1-\beta)k^2} \right\rceil$.

By calibrating the number of quadrature nodes with the number of basis functions in the FEM, an explicit rate of convergence for the strong error of the approximation $u^Q_{h,k} = Q^{\beta}_{h,k}\mathcal{W}_h$ was derived (Bolin et al., 2018, Theorem 2.10). Motivated by the asymptotic convergence of the method, it was suggested to choose $k \leq -\frac{\pi^2}{4\beta \ln(h)}$ in order to balance the errors induced by the quadrature and by a FEM of mesh size h (Bolin et al., 2018, Table 1). This corresponds to a total number of $K = K^- + K^+ + 1 > \frac{4\beta\ln(h)^2}{\pi^2(1-\beta)}$ quadrature nodes. The analogous result for the degree m of the approximation $u^R_{h,m}$ is given in Remark 3.4, suggesting the lower bound $m \geq \frac{\ln(h)^2}{\pi^2(1-\beta)}$, i.e., $K = 4\beta m$ asymptotically.

Furthermore, if we let $c_j := e^{2y_j}$ and

$$P^Q_{\ell,h} := \prod_{j=-K^-}^{K^+} c_j^{-\beta} \left(\mathrm{Id}_h + c_j L_h\right), \quad P^Q_{r,h} := \frac{2k\sin(\pi\beta)}{\pi} \sum_{i=-K^-}^{K^+} \prod_{j\neq i} c_j^{-\beta} \left(\mathrm{Id}_h + c_j L_h\right),$$

we find that the quadrature-based approximation $u^Q_{h,k}$ can equivalently be defined as the solution to the non-fractional SPDE

$$P^Q_{\ell,h} u^Q_{h,k} = P^Q_{r,h}\mathcal{W}_h \quad \text{in } \mathcal{D}. \tag{C.1}$$

Remark C.1. A comparison of (C.1) with (3.5) illustrates that $u^Q_{h,k}$ can be seen as a rational approximation of degree $K^- + K^+$, where the specific choice of the coefficients is implied by the quadrature. In combination with the remark above that $K = 4\beta m$ quadrature nodes are needed to balance the errors, this shows that the computational cost for achieving a given accuracy with the rational approximation from §3.3 is lower than with the quadrature method, since $\beta > d/4$.

Appendix D. Parameter identifiability

This section contains the proof of Theorem 6.1. For the proof, we will use the Feldman–Hájek theorem which we restate here from (Da Prato and Zabczyk, 2014, Theorem 2.25) for convenience.

Theorem D.1 (Felman–Hájek). *Two Gaussian measures $\mu_1 = \mathsf{N}(m_1, \mathcal{C}_1)$ and $\mu_2 = \mathsf{N}(m_2, \mathcal{C}_2)$ on a Hilbert space $\mathcal{H}$ are either singular or equivalent. They are equivalent if and only if the following three conditions are satisfied:*

I. $\mathrm{Im}\big(\mathcal{C}_1^{1/2}\big) = \mathrm{Im}\big(\mathcal{C}_2^{1/2}\big) := E$,
II. $m_1 - m_2 \in E$,
III. *the operator $T := \big(\mathcal{C}_1^{-1/2}\mathcal{C}_2^{1/2}\big)\big(\mathcal{C}_1^{-1/2}\mathcal{C}_2^{1/2}\big)^* - I$ is Hilbert-Schmidt in $\bar{E}$, where $*$ denotes the $\mathcal{H}$-adjoint operator, and I the identity on $\mathcal{H}$.*

Proof of Theorem 6.1. Since the two Gaussian measures have the same mean, we only have to verify conditions I. and III. of Theorem D.1.

We first prove that condition I. can hold only if $\beta_1 = \beta_2$. To this end, we use the equivalence of condition I. with the existence of two constants $c', c'' > 0$ such that

$$(v, \mathcal{C}_1 v)_{L_2(\mathcal{D})} \leq c'(v, \mathcal{C}_2 v)_{L_2(\mathcal{D})} \quad \text{and} \quad (v, \mathcal{C}_2 v)_{L_2(\mathcal{D})} \leq c''(v, \mathcal{C}_1 v)_{L_2(\mathcal{D})}, \tag{D.1}$$

see, e.g., (Stuart, 2010, Lemma 6.15), where in our case

$$\mathcal{C}_i := \mathcal{Q}_i^{-1} = \tau_i^{-2}(\kappa_i^2 - \Delta)^{-2\beta_i}, \qquad i \in \{1, 2\}.$$

Let λ_j^Δ, $j \in \mathbb{N}$, denote the positive eigenvalues (in nondecreasing order, counting multiplicity) of the Dirichlet or Neumann Laplacian $-\Delta\colon \mathscr{D}(\Delta) \to L_2(\mathcal{D})$, where the type of homogeneous boundary conditions is the same as for L_1 and L_2. By Weyl's law (3.3), there exist constants $\underline{c}, \bar{C} > 0$ such that

$$\underline{c}\, j^{2/d} \leq \lambda_j^\Delta \leq \bar{C} j^{2/d} \qquad \forall j \in \mathbb{N}.$$

Furthermore, we let $\{e_j\}_{j\in\mathbb{N}}$ denote a system of eigenfunctions corresponding to $\{\lambda_j^\Delta\}_{j\in\mathbb{N}}$ which is orthonormal in $L_2(\mathcal{D})$.

Now assume that $\beta_2 > \beta_1$ and let $j_0 \in \mathbb{N}$ be sufficiently large such that $\kappa_1^2 < \bar{C} j_0^{2/d}$. Then, we have

$$\frac{(\kappa_2^2 + \lambda_j^\Delta)^{2\beta_2}}{(\kappa_1^2 + \lambda_j^\Delta)^{2\beta_1}} > \frac{\underline{c}^{2\beta_2}}{(2\bar{C})^{2\beta_1}}\, j^{4(\beta_2-\beta_1)/d} \qquad \forall j \in \mathbb{N},\ j \geq j_0.$$

For any $N \in \mathbb{N}$, we can thus choose $j_* = j_*(N) \in \mathbb{N}$ sufficiently large such that

$$(e_{j_*}, \mathcal{C}_1 e_{j_*})_{L_2(\mathcal{D})} = \tau_1^{-2}(\kappa_1^2 + \lambda_{j_*}^\Delta)^{-2\beta_1} > N\tau_2^{-2}(\kappa_2^2 + \lambda_{j_*}^\Delta)^{-2\beta_2} = N(e_{j_*}, \mathcal{C}_2 e_{j_*})_{L_2(\mathcal{D})},$$

in contradiction with the first relation in (D.1), and μ_1, μ_2 are not equivalent if $\beta_1 \neq \beta_2$. Furthermore, condition I. is satisfied if $\beta_1 = \beta_2 = \beta > d/4$, since then, for all $v \in L_2(\mathcal{D})$,

$$\begin{aligned}
(v, \mathcal{C}_1 v)_{L_2(\mathcal{D})} &= \sum_{j\in\mathbb{N}} \tau_1^{-2}(\kappa_1^2 + \lambda_j^\Delta)^{-2\beta}(v, e_j)_{L_2(\mathcal{D})}^2 \\
&\leq \tau_1^{-2}\tau_2^2 \left(\min\left\{1, \kappa_1^2\kappa_2^{-2}\right\}\right)^{-2\beta} \sum_{j\in\mathbb{N}} \tau_2^{-2}(\kappa_2^2 + \lambda_j^\Delta)^{-2\beta}(v, e_j)_{L_2(\mathcal{D})}^2 \\
&= \tau_1^{-2}\tau_2^2 \max\left\{1, \kappa_1^{-4\beta}\kappa_2^{4\beta}\right\}(v, \mathcal{C}_2 v)_{L_2(\mathcal{D})},
\end{aligned}$$

and, similarly, $(v, \mathcal{C}_2 v)_{L_2(\mathcal{D})} \leq \tau_2^{-2}\tau_1^2 \max\{1, \kappa_2^{-4\beta}\kappa_1^{4\beta}\}(v, \mathcal{C}_1 v)_{L_2(\mathcal{D})}$. Thus, (D.1) and condition I. of Theorem D.1 hold.

Assuming that $\beta_1 = \beta_2 = \beta > d/4$, it remains now to show that condition III. of Theorem D.1 is satisfied if and only if $\tau_1 = \tau_2$. To this end, we first note that the operator $T := \mathcal{C}_1^{-1/2}\mathcal{C}_2\mathcal{C}_1^{-1/2} - I$ has eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$ and eigenvalues

$$\tau_1^2\tau_2^{-2}(\kappa_1^2 + \lambda_j^\Delta)^{2\beta}(\kappa_2^2 + \lambda_j^\Delta)^{-2\beta} - 1, \qquad j \in \mathbb{N}.$$

Therefore, T is Hilbert–Schmidt in $\bar{E}$ if and only if

$$\sum_{j\in\mathbb{N}} \left(\tau_1^2\tau_2^{-2}(\kappa_1^2 + \lambda_j^\Delta)^{2\beta}(\kappa_2^2 + \lambda_j^\Delta)^{-2\beta} - 1\right)^2 < \infty. \tag{D.2}$$

Since $x \mapsto (1 + x)^{1/(2\beta)}$ is monotonically increasing in $x > 0$, again by the Weyl asymptotic, for any $\varepsilon_0 > 0$, we can find an index $j_0 \in \mathbb{N}$ such that

$$\frac{\kappa_2^2}{\lambda_j^\Delta} + 1 \leq (1 + \varepsilon_0)^{1/(2\beta)} \qquad \forall j \in \mathbb{N},\ j \geq j_0. \tag{D.3}$$

Assume that $\tau_1 \neq \tau_2$ and without loss of generality let $\tau_1 > \tau_2$. Then pick $\varepsilon_0 > 0$ such that $\tau_1^2\tau_2^{-2} \geq 1 + 2\varepsilon_0$, and $j_0 \in \mathbb{N}$ such that (D.3) holds. These choices give

$$\tau_1^2\tau_2^{-2}\left(\frac{\kappa_1^2+\lambda_j^\Delta}{\kappa_2^2+\lambda_j^\Delta}\right)^{2\beta} \geq \tau_1^2\tau_2^{-2}(\kappa_2^2/\lambda_j^\Delta+1)^{-2\beta} \geq (1+2\varepsilon_0)(1+\varepsilon_0)^{-1} > 1,$$

for all $j \in \mathbb{N}$ with $j \geq j_0$. Thus, the series in (D.2) is unbounded,

$$\begin{aligned}\sum_{j\in\mathbb{N}} \left(\tau_1^2\tau_2^{-2}(\kappa_1^2+\lambda_j^\Delta)^{2\beta}(\kappa_2^2+\lambda_j^\Delta)^{-2\beta}-1\right)^2 &\geq \sum_{j\geq j_0}\left((1+2\varepsilon_0)(1+\varepsilon_0)^{-1}-1\right)^2 \\ &= \sum_{j\geq j_0} \varepsilon_0^2\,(1+\varepsilon_0)^{-2} = \infty.\end{aligned}$$

We conclude that condition III. of Theorem D.1 is not satisfied if $\tau_1 \neq \tau_2$.

Finally, let $\beta_1 = \beta_2 = \beta$, $\tau_1 = \tau_2$ and assume without loss of generality that $\kappa_2 > \kappa_1$ (if $\kappa_1 = \kappa_2$, (D.2) is evident). By the mean value theorem, applied for the function $x \mapsto x^{2\beta}$, for every $j \in \mathbb{N}$, there exists $\widetilde{\kappa}_j \in (\kappa_1, \kappa_2)$ such that

$$(\kappa_2^2+\lambda_j^\Delta)^{2\beta} - (\kappa_1^2+\lambda_j^\Delta)^{2\beta} = 2\beta(\widetilde{\kappa}_j^2+\lambda_j^\Delta)^{2\beta-1}(\kappa_2^2-\kappa_1^2).$$

Hence, we can bound the series in (D.2) as follows,

$$\begin{aligned}\sum_{j\in\mathbb{N}}\left(\frac{(\kappa_1^2+\lambda_j^\Delta)^{2\beta}-(\kappa_2^2+\lambda_j^\Delta)^{2\beta}}{(\kappa_2^2+\lambda_j^\Delta)^{2\beta}}\right)^2 &= 4\beta^2(\kappa_2^2-\kappa_1^2)^2\sum_{j\in\mathbb{N}}\left(\frac{(\widetilde{\kappa}_j^2+\lambda_j^\Delta)^{2\beta-1}}{(\kappa_2^2+\lambda_j^\Delta)^{2\beta}}\right)^2 \\ \leq 4\beta^2(\kappa_2^2-\kappa_1^2)^2\sum_{j\in\mathbb{N}}(\widetilde{\kappa}_j^2+\lambda_j^\Delta)^{-2} &\leq 4\beta^2(\kappa_2^2-\kappa_1^2)^2\underline{c}^{-2}\sum_{j\in\mathbb{N}} j^{-4/d} < \infty.\end{aligned}$$

Here, $\sum_{j\in\mathbb{N}} j^{-4/d}$ converges, since $4/d > 1$ for $d \in \{1,2,3\}$. This proves equivalence of the Gaussian measures if $\beta_1 = \beta_2$ and $\tau_1 = \tau_2$. □

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