Supplemental Web Material: Bayesian Hierarchical Multi-Population Multistate Jolly-Seber Models with Covariates: Application to the Pallid Sturgeon Population Assessment Program

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Appendix A: Full Conditional Distributions

We provide details about the full conditional distributions for the multi-population multistate Jolly-Seber (MP-MSJS) model with covariates using state-space modeling.

- $[N^k|\cdot]$, $k = 1, 2, \ldots, K$.

$$[N^k|\cdot] \propto \binom{N^k}{n^k} \left[1 - \sum_{t=1}^{T} \sum_{s=1}^{G} \alpha_t(s) p_t^k(s)\right]^{N^k - n^k} \pi(N^k) 1(N^k \geq n^k),$$

where $\alpha_1^k(s) = \beta_0^k(s)$ and for $t = 1, 2, \ldots, T-1$ the recursion follows that

$$\alpha_{t+1}^k(s) = \beta_t^k(s) + \sum_{r=1}^{G} \alpha_t^k(r) \left[1 - p_t^k(r)\right] q_t^k(r, s), \ s \in \mathcal{G}.$$

Let $\lambda^k = \sum_{t=1}^{T} \sum_{r=1}^{G} \alpha_t^k(r) p_t^k(r)$ refer to the probability for individuals of population $\mathcal{P}^k$ in the study area to be marked, it then follows

$$N^k - n^k|\cdot \sim \begin{cases} \text{NegBin}(n^k, \lambda^k), & \text{if } \pi(N^h) \propto \frac{1}{N^k} \\ \text{NegBin}(n^k + 1, \lambda^k), & \text{if } \pi(N^k) \propto 1 \end{cases}$$

where $\text{NegBin}(n, p)$ denotes the negative binomial distribution with probability mass function being

$$p_X(x) = \binom{n + x - 1}{x} p^n (1 - \lambda)^x, x = 0, 1, \ldots, n > 0 \text{ and } 0 < p \leq 1$$

- $[\boldsymbol{\psi}_t^k(r, \cdot)|\cdot]$, $r \in \mathcal{G}$ and $k = 1, 2, \ldots, K$.

$$[\boldsymbol{\psi}_t^k(r, \cdot)|\cdot] \propto \prod_{s=1}^{G} \psi_t^k(r, s)^{W_t^k(r,s)}.$$

where $W_t^k(r, s) = \sum_{i=1}^{N^k} z_{i,t+1}^k(s) z_{i,t}^k(r)$ for $s \in \mathcal{G}$. Therefore,

$$\boldsymbol{\psi}_t^k(r, \cdot)|\cdot \sim \text{Dirichlet}\left(e_t^k(r, 1) + W_t^k(r, 1), \ldots, e_t^k(r, G) + W_t^k(r, G)\right).$$

- $[\boldsymbol{\beta}^k|\cdot]$, $r \in \mathcal{G}$, $t = 0, \ldots, T-1$ and $k = 1, 2, \ldots, K$.

$$[\boldsymbol{\beta}^k|\cdot] \propto \prod_{t=0}^{T-1} \prod_{r=1}^{G} \left\{ \beta_t^k(r)^{E_t^k(r)} \right\} [\boldsymbol{\beta}],$$

i.e.,

$$\boldsymbol{\beta}^k \sim \text{Dirichlet}(\gamma_t^k(r) + E_t^k(r); r \in \mathcal{G}, t = 0, \ldots, T-1).$$

where $E_t^k(r,s) = \sum_{i=1}^{N^k} 1\left(z_{i,t+1}^k = r\right) 1\left(z_{i,t}^k = \star\right)$

- $[\theta_p|\cdot]$, $p = 1, 2, \ldots, P$.

$$\theta_p|\cdot \sim N(\mu_{\theta_p|\cdot}, \sigma^2_{\theta_p|\cdot}),$$

where

$$\sigma^2_{\theta_p|\cdot} = \left\{ \sum_{k=1}^{K} \frac{\sum_{t=1}^{T} \sum_{r=1}^{G} x_{tr,p}^2}{\sigma_k^2} + \frac{1}{\sigma^2_{\theta_p}} \right\}^{-1},$$

$$\mu_{\theta_p|\cdot} = \sigma^2_{\theta_p|\cdot} \left\{ \sum_{k=1}^{K} \frac{\sum_{t=1}^{T} \sum_{r=1}^{G} x_{tr,p}(\widetilde{p}_t^k(r) - \sum_{s \neq p} x_{tr,s}\theta_s)}{\sigma_k^2} + \frac{\mu_{\theta_p}}{\sigma^2_{\theta_p}} \right\}.$$

- $\sigma_k^2|\cdot \sim \text{IG}(C_k^*, D_k^*)$, $k = 1, 2, \ldots, K$ and

$$C_k^* = \frac{TG}{2} + C_k,$$

$$D_k^* = \frac{\sum_{t=1}^{T} \sum_{r=1}^{G} (\widetilde{p}_t^k(r) - \mathbf{X}'_{tr}\boldsymbol{\theta})^2}{2} + D_k.$$

- $\gamma_m|\cdot \sim N(\mu_{\gamma_m|\cdot}, \sigma^2_{\gamma_m|\cdot})$, $m = 1, 2, \ldots, M$, where

$$\sigma^2_{\gamma_m|\cdot} = \left\{ \sum_{k=1}^{K} \frac{\sum_{t=1}^{T-1} \sum_{r=1}^{G} \Omega_{tr,m}^2}{\xi_k^2} + \frac{1}{\sigma^2_{\gamma_m}} \right\}^{-1},$$

$$\mu_{\gamma_m|\cdot} = \sigma^2_{\gamma_m|\cdot} \left\{ \sum_{k=1}^{K} \frac{\sum_{t=1}^{T-1} \sum_{r=1}^{G} \Omega_{tr,m}(\widetilde{\phi}_t^k(r) - \sum_{s \neq k} \Omega_{tr,s}\gamma_s)}{\xi_k^2} + \frac{\mu_{\gamma_m}}{\sigma^2_{\gamma_m}} \right\}.$$

- $\xi_k^2|\cdot \sim \text{IG}(A_k^*, B_k^*)$, $k = 1, 2, \ldots, K$ and

$$A_k^* = \frac{(T-1)G}{2} + A_k,$$

$$B_k^* = \frac{\sum_{t=1}^{T-1} \sum_{r=1}^{G} (\widetilde{\phi}_t^k(r) - \boldsymbol{\Omega}'_{tr}\boldsymbol{\gamma})^2}{2} + B_k.$$

- $[\widetilde{p}_t^k(r)|\cdot]$, $t = 1, 2, \ldots, T$, $k = 1, 2, \ldots, K$ and $r \in \mathcal{G}$

$$[\widetilde{p}_t^k(r)|\cdot] \propto \left\{\exp(\widetilde{p}_t^k(r))\right\}^{U_t^k(r)} \left\{1 + \exp(\widetilde{p}_t^k(r))\right\}^{-N_t^k(r)} \exp\left\{-\frac{(\widetilde{p}_t^k(r) - \mathbf{X}_{tr}'\boldsymbol{\theta})^2}{2\sigma_k^2}\right\}.$$

- $[\widetilde{\phi}_t^k(r)|\cdot]$, $t = 1, 2, \ldots, T-1$, $k = 1, 2, \ldots, K$ and $r \in \mathcal{G}$

$$[\widetilde{\phi}_t^k(r)|\cdot] \propto \left\{\exp(\widetilde{\phi}_t^k(r))\right\}^{W_t^k(r)} \left\{1 + \exp(\widetilde{\phi}_t^k(r))\right\}^{-N_t^k(r)} \exp\left\{-\frac{(\widetilde{\phi}_t^k(r) - \boldsymbol{\Omega}_{tr}'\boldsymbol{\gamma})^2}{2\xi_k^2}\right\}.$$

Appendix B: Proof of Lemma 3.1

We can easily show that the full conditionals of $\widetilde{p}_t^k(r)$ and $\widetilde{\phi}_t^k(r)$ take the following forms

$$[\widetilde{p}_t^k(r)|\cdot] \propto \left\{\exp(\widetilde{p}_t^k(r))\right\}^{U_t^k(r)} \left\{1 + \exp(\widetilde{p}_t^k(r))\right\}^{-N_t^k(r)} \exp\left\{-\frac{(\widetilde{p}_t^k(r) - \mathbf{X}_{tr}'\boldsymbol{\theta})^2}{2\sigma_k^2}\right\}, \tag{B.1}$$

$$[\widetilde{\phi}_t^k(r)|\cdot] \propto \left\{\exp(\widetilde{\phi}_t^k(r))\right\}^{W_t^k(r)} \left\{1 + \exp(\widetilde{\phi}_t^k(r))\right\}^{-N_t^k(r)} \exp\left\{-\frac{(\widetilde{\phi}_t^k(r) - \boldsymbol{\Omega}_{tr}'\boldsymbol{\gamma})^2}{2\xi_k^2}\right\}. \tag{B.2}$$

where

$$\begin{aligned}
U_t^k(r) &= \sum_{i=1}^{n^k} y_{i,t}^k(r),\ \ t = 1, 2, \ldots, T, \\
W_t^k(r) &= \sum_{i=1}^{N^k}\sum_{s=1}^{G} 1(z_{i,t+1}^k = s)1(z_{i,t}^k = r),\ \ t = 1, 2, \ldots, T-1, \\
N_t^k(r) &= \sum_{i=1}^{N^k} 1(z_{i,t}^k = r),\ \ t = 1, 2, \ldots, T,
\end{aligned}$$

for $k = 1, 2, \ldots, K$ and $r \in \mathcal{G}$. In other words, $U_t^k(r)$ is the total number of individuals from population $\mathcal{P}^k$ that are captured at location r at time t; $W_t^k(r)$ is the total number of individuals from population $\mathcal{P}^k$ present at region r at time t that survive to time $t+1$; and $N_t^k(r)$ is the total number of individuals from population $\mathcal{P}^k$ that are alive and present at region r at time t. Moreover, n^k is the total number of individuals caught from population $\mathcal{P}^k$.

Taking the logarithm of (B.1) and (B.2) yields

$$\log[\widetilde{p}_t^k(r)|\cdot] = \text{const} + U_t^k(r)\widetilde{p}_t^k(r) - N_t^k(r)\log\left\{1 + \exp(\widetilde{p}_t^k(r))\right\} - \frac{(\widetilde{p}_t^k(r) - \mathbf{X}'_{tr}\boldsymbol{\theta})^2}{2\sigma_k^2},$$

$$\log[\widetilde{\phi}_t^k(r)|\cdot] = \text{const} + W_t^k(r)\widetilde{\phi}_t^k(r) - N_t^k(r)\log\left\{1 + \exp(\widetilde{\phi}_t^k(r))\right\} - \frac{(\widetilde{\phi}_t^k(r) - \boldsymbol{\Omega}'_{tr}\boldsymbol{\gamma})^2}{2\xi_k^2},$$

where `const` denotes any constant.

The derivation of the first and the second derivatives of $\log[\widetilde{p}_t^k(r)|\cdot]$ with respect to $\widetilde{p}_t^k(r)$ yields

$$\frac{\partial\log[\widetilde{p}_t^k(r)|\cdot]}{\partial\widetilde{p}_t^k(r)} = U_t^k(r) - N_t^k(r)\frac{\exp\{\widetilde{p}_t^k(r)\}}{1 + \exp\{\widetilde{p}_t^k(r)\}} - \frac{\widetilde{p}_t^k(r) - \boldsymbol{X}'_{tr}\boldsymbol{\theta}}{\sigma_k^2},$$

$$\frac{\partial^2\log[\widetilde{p}_t^k(r)|\cdot]}{\partial\{\widetilde{p}_t^k(r)\}^2} = -N_t^k(r)\frac{\exp\{\widetilde{p}_t^k(r)\}}{\{1 + \exp\{\widetilde{p}_t^k(r)\}\}^2} - \frac{1}{\sigma_k^2} < 0.$$

Similarly for $\widetilde{\phi}_t^k(r)$, we have

$$\frac{\partial\log[\widetilde{\phi}_t^k(r)|\cdot]}{\partial\ \widetilde{\phi}_t^k(r)} = W_t^k(r) - N_t^k(r)\frac{\exp\{\widetilde{\phi}_t^k(r)\}}{1 + \exp\{\widetilde{\phi}_t^k(r)\}} - \frac{\widetilde{\phi}_t^k(r) - \boldsymbol{\Omega}'_{tr}\boldsymbol{\gamma}}{\xi_k^2},$$

$$\frac{\partial^2\log[\widetilde{\phi}_t^k(r)|\cdot]}{\partial\ \{\widetilde{\phi}_t^k(r)\}^2} = -N_t^k(r)\frac{\exp\{\widetilde{\phi}_t^k(r)\}}{\{1 + \exp\{\widetilde{\phi}_t^k(r)\}\}^2} - \frac{1}{\xi_k^2} < 0.$$

As a result, the full conditionals of $\widetilde{p}_t^k(r)$ and $\widetilde{\phi}_t^k(r)$ are log-concave.

Appendix C: Sampling $\boldsymbol{z}$

To update the latent variables $\boldsymbol{z}_i$, we adapt the sampling algorithm developed by Dupuis and Schwarz (2007). For completeness, herein, we describe the algorithm in detail. Without loss of generality, we consider the single species case with $\mathcal{G} = \{1, 2, 3\}$ and $T = 10$, since the algorithm for the latent variables remains the same within each population. Let N denote the population size. For $i = 1, 2, \ldots, N$, we define two sets as follows:

$$S_1 = \{i : \boldsymbol{y}_i \neq \boldsymbol{0}\},$$

$$S_2 = \{i : \boldsymbol{y}_i = \boldsymbol{0}\},$$

where $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,T})$ denotes the capture history for an individual i. In other words, the set S_1 contains individuals that are captured at least once during T sampling occasions; whereas the set S_2 includes individuals that are never captured during the study. The update of latent variables $\boldsymbol{z}_i$ will depend on the set that an individual i falls into. For example, consider a capture history of the form

$$\boldsymbol{y}_i = 0010030020, \tag{C.1}$$

which indicates that an individual i is captured in location 1 at $t = 3$, in location 3 at $t = 6$, and last captured in location 2 at $t = 9$. It follows from (C.1) that the corresponding $\boldsymbol{z}_i$ takes the form of $\boldsymbol{z}_i = \cdot\cdot 1 \cdot\cdot 3 \cdot\cdot 2\cdot$, where $\cdot$ denotes a missing location that needs to be simulated.

Before we move on to the sampling algorithm, we need to introduce some notation. First, we define f_i and l_i as the first and last times that an individual $i \in S_1$ is captured. Moreover, define e_i as the time that individual i first enters the study area, which is unknown. Furthermore, three types of blocks due to Dupuis and Schwarz (2007) need to be in place. First, the Type I block $B_1(i)$ consists of state variables corresponding to sampling times up to f_i. Second, the Type II block $B_2(i)$ includes state variables corresponding to two successive capture events. Third, the Type III block $B_3(i)$ is comprised of state variables corresponding to sampling occasions right after l_i to time T. Hence, for the capture history given in (C.1), it follows

$$\begin{aligned}
B_1(i) &= \{z_{i,1}, z_{i,2}, z_{i,3}\}, \\
B_2(i) &= \{(z_{i,e_i}, z_{i,3}), (z_{i,3}, z_{i,6}), (z_{i,6}, z_{i,9})\}, \\
B_3(i) &= \{z_{i,10}\},
\end{aligned}$$

where e_i denotes the time that individual i first enters the study area and it is unknown.

For convenience of exposition, define $m_{t_1,s}^{t_2,r}$ as the probability that an individual moves from location s at time t_1 to location r at time t_2, where $t_2 > t_1$. The following recursive relationship

$$
\begin{aligned}
m_{t,s}^{t+1,r} &= \phi_t(s)\psi_t(s,r), \text{ for } t_1 \le t < t_2, \\
m_{t,s}^{t+2,r} &= \sum_{s'=1}^{G} m_{t,s}^{t+1,s'} m_{t+1,s'}^{t+2,r}, \text{ for } t_1 \le t < t_2 - 1,
\end{aligned}
$$

is needed to derive $m_{t_1,s}^{t_2,r}$ for $s, r \in \mathcal{G}$ and $1 \le t_1 < t_2 \le T$.

We now elaborate on the updating scheme of $\boldsymbol{z}_i$ for $i \in S_1$. For $t = 1, 2, \ldots, T-1$ and $r \in \mathcal{G}$, define $\lambda_t(r)$ as

$$
\lambda_{t+1}(r) = \beta_t(r) + \sum_{s=1}^{G} \lambda_t(s)\{1 - p_t(s)\}\phi_t(s)\psi_t(s,r),
$$

and $\lambda_1(r) = \beta_0(r)$. For Type I block simulation, we need to determine e_i and z_{i,e_i}, which can be achieved by first simulating $\boldsymbol{u}_i \sim \text{Multinomial}(1, \boldsymbol{\xi}_i)$ where

$$
\begin{aligned}
\boldsymbol{u}_i &= (u_{i,1}(1), \ldots, u_{i,1}(G), u_{i,2}(1), \ldots, u_{i,2}(G), \ldots, u_{i,f_i}(1), \ldots, u_{i,f_i}(G)), \\
\boldsymbol{\xi}_i &= (\xi_{i,1}(1), \ldots, \xi_{i,1}(G), \xi_{i,2}(1), \ldots, \xi_{i,2}(G), \ldots, \xi_{i,f_i}(1), \ldots, \xi_{i,f_i}(G)), \\
\xi_{i,t}(s) &= \frac{\beta_{t-1}(s)\{1 - p_t(s)\} m_{t,s}^{f_i,r}}{\lambda_{f_i}(r)}, \ r, \ s \in \mathcal{G} \text{ and } t = 1, 2, \ldots, f_i.
\end{aligned}
$$

As a result, we can determine e_i and z_{i,e_i} according to

$$
\begin{aligned}
e_i &= \mathcal{T}(\boldsymbol{u}_i), \\
z_{i,e_i} &= \mathcal{S}(\boldsymbol{u}_i).
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}(\boldsymbol{u}_i) &= 1 + \left\lfloor \frac{\text{Ind}(\boldsymbol{u}_i) - 1}{G} \right\rfloor, \\
\mathcal{S}(\boldsymbol{u}_i) &= \begin{cases} G, & \text{if } (\text{Ind}(\boldsymbol{u}_i) \bmod G) = 0, \\ \text{Ind}(\boldsymbol{u}_i) \bmod G, & \text{otherwise,} \end{cases}
\end{aligned}
$$

where for a standard unit vector $\boldsymbol{x}$ in $\mathbf{R}^d$, $\mathrm{Ind}(\boldsymbol{x}) = j$ if $x_j = 1$ for $1 \le j \le d$. Moreover, $\lfloor u \rfloor$ denotes the largest integer not greater than u. After e_i is determined, we then set $z_{i,t} = \star$ for $1 \le t < e_i$.

For $(z_{i,t_1}, z_{i,t_2}) \in B_2(i)$, we need to simulate latent states between two consecutive "capture occasions" t_1 and t_2. For the trivial case where $t_1 = t_2 - 1$, there is no need to update the latent state variable. As a result, we assume $z_{i,t_1} = s$ and $z_{i,t_2} = r$ for $1 \le t_1 < t_2 \le T$ and $r, s \in \mathcal{G}$. Our goal is to simulate missing state variables $z_{i,t}$ for $t_1 < t < t_2$. To this end, we define

$$\begin{aligned}\alpha_{j,s}^{j',r}(s') &= P(z_{i,j+1} = s' | z_{i,j} = s, z_{i,j+1} = \cdot, \ldots, z_{i,j'-1} = \cdot, z_{i,j'} = r) \\ &= \frac{\phi_j(s)\psi_j(s,s')\{1 - p_{j+1}(s')\} m_{j+1,s'}^{j',r}}{m_{j,s}^{j',r}},\end{aligned}$$

for $j < j'$. Subsequently, we simulate $\boldsymbol{v}_i \sim \mathrm{Multinomial}(1, \boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = (\alpha_{j,1}^{t_2,r}, \alpha_{j,2}^{t_2,r}, \ldots, \alpha_{j,G}^{t_2,r})$. Then, we set $z_{i,j+1} = \mathrm{Ind}(\boldsymbol{v}_i)$ for $t_1 \le j < t_2 - 1$.

Next, we discuss the simulation for the latent variables in the Type III block. For $1 \le t < T$ and $s \in \mathcal{G}$, let $w_t(s)$ denote the probability that an individual leaves the study area after time t at location s. We can obtain $w_t(s)$ using the recursion

$$w_t(s) = 1 - \phi_t(s) + \phi_t(s) \sum_{s'=1}^{G} \psi_t(s,s')\{1 - p_{t+1}(s')\} w_{t+1}(s'), \; t = T-1, T-2, \ldots, 1,$$

and $w_T(s) = 1$. To update $z_{i,t} \in B_3(i)$ for $t = l_i + 1, \ldots, T$, we first simulate $o_{i,t}$ from

$$o_{i,t}(s) \sim \mathrm{Bernoulli}\left(\frac{1 - \phi_{t-1}(s)}{w_{t-1}(s)}\right),$$

and then determine $z_{i,t}$ according to

$$z_{i,t} = \begin{cases} \dagger, & \text{if } z_{i,t-1} = \dagger \text{ or } o_{i,t}(s) = 1, \\ \mathrm{Ind}(\boldsymbol{\tau}_{i,t}), & \text{otherwise}, \end{cases} \tag{C.2}$$

where $\boldsymbol{\tau}_{i,t} \sim \mathrm{Multinomial}(1, \boldsymbol{\varsigma}_t)$ with $\boldsymbol{\varsigma}_t = (\varsigma_t(1), \ldots, \varsigma_t(G))$ and

$$\varsigma_t(r) = \frac{\phi_t(s)\psi_t(s,r)\{1 - p_{t+1}(r)\} w_{t+1}(r)}{w_t(s)}, \text{ for } r \in \mathcal{G},$$

assuming $z_{i,t-1} = s \in \mathcal{G}$.

Lastly, we address the simulation of the latent variables $\boldsymbol{z}_i \in S_2$, i.e., for an individual that is never captured during the entire study. This requires us to first determine the time that an individual first enters the population (i.e., Type I block simulation) followed by ascertaining the status at subsequent sampling occasions after its entrance (i.e., Type III block simulation). To this end, we define ρ as the probability that an individual is never captured. Then, we can derive the following

$$\rho = 1 - \sum_{t=1}^{T} \sum_{r=1}^{G} \lambda_t(r) p_t(r).$$

To determine e_i for an individual i, we simulate $\boldsymbol{\zeta}_i \sim \text{Multinomial}(1, \boldsymbol{\varpi})$ where

$$\begin{aligned}
\boldsymbol{\zeta} &= (\zeta_1(1), \ldots, \zeta_1(G), \zeta_2(1), \ldots, \zeta_2(G), \ldots, \zeta_T(1), \ldots, \zeta_T(G)), \\
\boldsymbol{\varpi} &= (\varpi_1(1), \ldots, \varpi_1(G), \varpi_2(1), \ldots, \varpi_2(G), \ldots, \varpi_T(1), \ldots, \varpi_T(G)), \\
\varpi_t(s) &= \frac{\beta_{t-1}(s)\{1 - p_t(s)\} w_t(s)}{\rho}, \; s \in \mathcal{G} \text{ and } t = 1, 2, \ldots, T.
\end{aligned}$$

Then we can determine e_i and z_{i,e_i} according to

$$\begin{aligned}
e_i &= \mathcal{T}(\boldsymbol{\zeta}), \\
z_{i,e_i} &= \mathcal{S}(\boldsymbol{\zeta}).
\end{aligned}$$

After e_i and z_{i,e_i} are determined, we need to perform the Type III block simulation for S_2. The details are omitted here due to its similarity with the Type III block simulation for S_1 in the previous discussion.

Appendix D: Likelihood function for complete data

The likelihood function for complete data $L(\boldsymbol{Y}, \boldsymbol{z}|\boldsymbol{\Theta})$ has the following form

$$\begin{aligned}
L(\boldsymbol{Y}, \boldsymbol{z}|\boldsymbol{\Theta}) &= \prod_{k=1}^{K} L_{\phi,\psi}^{k} \times L_{\beta}^{k} \times L_{p}^{k} \\
&= \prod_{k=1}^{K} \left\{ \prod_{t=0}^{T-1} \prod_{r\in\mathcal{G}} \left(\beta_t^k(r)\right)^{E_t^k(r)} \right\} \\
&\times \left\{ \prod_{t=1}^{T-1} \prod_{r\in\mathcal{G}} \left(\phi_t^k(r)\right)^{W_t^k(r)} \left(1-\phi_t^k(r)\right)^{N_t^k(r)-W_t^k(r)} \prod_{s\in\mathcal{G}} \left(\psi_t^k(r,s)\right)^{W_t^k(r,s)} \right\} \\
&\times \prod_{t=1}^{T} \prod_{r\in\mathcal{G}} \left(p_t^k(r)\right)^{U_t^k(r)} \left(1-p_t^k(r)\right)^{N_t^k(r)-U_t^k(r)}.
\end{aligned}$$

The log-likelihood function for complete data $\ell(\boldsymbol{Y}, \boldsymbol{z}|\boldsymbol{\Theta}) = \ln L(\boldsymbol{Y}, \boldsymbol{z}|\boldsymbol{\Theta})$.

References

Dupuis, J. A. and Schwarz, C. J. (2007). "A Bayesian approach to the multistate Jolly–Seber capture–recapture model." *Biometrics*, 63, 4, 1015–1022.